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Optimizing Multi-attributive Decision-making: A Novel Approach through Matrix Theory and Fuzzy Hypersoft Sets

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ABSTRACT

Multi-criteria decision-making (MCDM) focuses on managing and prioritizing decisions that encompass multiple criteria. The use of fuzzy soft set frameworks is limited in addressing certain problems when multiple and subdivided attributes are involved. Consequently, there was a pressing demand for a novel methodology capable of overcoming these challenges. To this end, the concept of fuzzy hypersoft matrix (FHSM) is developed. This paper introduces various principles related to FHSM, including operations like union, intersection, subsets, equality, complements, empty sets, and universal sets. It provides numerous apt examples to validate the defined notions effectively. Additionally, the paper describes the application of FHSM in creating a system for recognizing objects from vague data across multiple observers.

1. Introduction

Making decisions and solving problems is one of the most difficult aspects of our lives. As a result, we must prioritize the best multiple options. Multi-attribute decision aids us in making a decision in this case. However, it is possible to gather unreliable information while making a decision. Decisions involving uncertainty must be communicated at various stages of life in order to overcome real-life obstacles. Uncertainty, ambiguity, and unreliability in data are the most important factors to consider when dealing with these issues.

Various mathematical theories have been introduced to address these issues, including probability theory, fuzzy set theory [1], and rough set theory [2]. Zadeh's introduction of fuzzy set theory has gained significant popularity in addressing uncertainty concerns. This theory provides an appropriate framework for describing uncertain notions by allowing for the use of partial membership functions. Mathematicians and computer scientists have researched and developed fuzzy sets, leading to the discovery of several practical applications such as fuzzy control systems, fuzzy automata, fuzzy logic, and fuzzy topology. Molodtsov [3] introduced soft set theory in 1999 as

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a novel technique for representing uncertainty, addressing specific structural challenges of fuzzy set theory and other theories.

Many researchers have made significant attempts to generalize and extend the concept of soft sets proposed by Molodtsov [3]. Maji et al. [4] have developed a hybrid structure called the fuzzy soft cluster structure by combining the fuzzy set and soft set structure. Put simply, while establishing a fuzzy soft set, a degree is included in the parameterization of fuzzy sets [5]. The fuzzy soft set structure, which is a synthesis of the soft set structure and fuzzy set structure, has been extensively employed by researchers, with numerous papers contributing to the existing literature [6-8].

Researchers' great interest in this area has led to significant advancements in the application of fuzzy soft set structure in decision-making challenges. Unrestricted definition of unreal objects in soft sets allows researchers to select the desired parameter format, hence streamlining the decision-making process and enhancing efficiency in situations where some information is absent. Maji & Roy [9] were the first to utilize soft sets in decision-making problems. Chen [10] explained the process of simplifying the parameterization of the soft set and examined its use in the decision-making problem. Cagman & Enginoglu [11-12] examined the concept of soft matrix and uni-int decision-making, which involved selecting the most favorable elements from a range of possibilities.

This work presents a clear definition of fundamental concepts, including subset, equal set, union, intersection, complement, null set, and absolute set, as well as the AND and OR operations on the fuzzy hypersoft set structure [13-14]. In addition, we utilized fuzzy hypersoft sets to address the decision-making issue. By utilizing Roy & Maji's technique [15], we have formulated a suitable choice problem for fuzzy hypersoft sets. This paper introduces the fuzzy hypersoft set structure as a fundamental characteristic. Thus, it plays a crucial part in numerous following investigations [16-17]. Extensive research has been conducted in the literature on decision-making challenges [18], with numerous researchers investigating this topic [19]. The user's text is incomplete and cannot be rewritten in a straightforward and precise manner [20-21]. Smarandache [22] introduced a novel methodology for managing uncertainty. He extended the soft set to a hypersoft set by converting the functionality into a multi-decision function. Despite being a more recent development, the hypersoft set theory has garnered significant interest from scholars, as seen by studies conducted by researchers [23-24].

The concepts of linguistic hypersoft set and fuzzy linguistic hypersoft set have been proposed by [25]. Some more optimization and decision-making approaches have been used to solve optimization problems [26]. The machine learning tools along with decision-making algorithms have been employed in many real-world examples [27-28]. Various intricate issues arise from the presence of ambiguous data in disciplines such as social sciences, economics, medical sciences, engineering, and other fields. These challenges encountered in life cannot be resolved using conventional mathematical tools. In classical mathematics, a model is constructed with precision and accuracy [29-30].

This research improves the robustness, flexibility, and usefulness of multi-criteria decision-making (MCDM) strategies in traversing the complexities of real-world problems by filling in the gaps in current methodology and introducing new ideas and tools.

i. This technique is unique because it combines the structural depth of hypersoft sets with the elasticity of fuzzy systems. It is designed specifically to handle ambiguous information and multiple criterion and decision variables. Traditional MCDM techniques encounter numerous challenges due to the inherent ambiguity of real-world data. The fuzzy hypersoft matrix (FHSM) provides a method that effectively captures and analyzes the inherent uncertainty present in established processes, especially in complex and dynamic situations.

ii. This study has significant implications for the realm of decision-making. Firstly, it presents a consolidated framework that adeptly manages several factors and the lack of clarity in the data to establish a more precise and dependable basis for decision-making. Furthermore, it broadens the scope of MCDM approaches, allowing for their application in a wider range of scenarios, particularly those where imprecise, ambiguous, or missing information has previously impeded their use. Finally, this study offers relevant information by bridging the divide between theoretical models and their practical applications. It accomplishes this by demonstrating the utilization of the Fuzzy hybrid scatter search method in several settings.

iii. Although the stated topics have made progress, there is still a significant research gap in integrating fuzzy logic and MCDM approaches with hypersoft sets. Prior research has focused on the incorporation of more advanced mathematical prerequisites in HSM or has investigated the possibility of using fuzzy logic as a separate technique. The lack of thorough research on the integration of hypersoft matrix (HSM) with other subjects is evident, and it perpetuates the existing gap in knowledge when examining MCDM. FHSM is a methodology that enhances decision-making in various areas that rely on intricate criteria, ambiguous data, or both. It has the potential for wider application in fields like engineering, healthcare, environmental management, and policymaking.

The organization of the research paper is structured in the following manner: Section 2 provides preliminaries. Section 3 presents the definition and operations of hypersoft matrix theory. Section 4 provides the MCDM algorithm. Section 5 gives a case study. Section 6 discusses the results and compares them with existing studies. Finally, the findings of the study and their implications with possible future directions are presented in Section 7.

2. Preliminaries

2.1. Soft Sets

Let us consider that E is the attributive set and $\mathbb{U}$ is the universal set. The P($\mathbb{U}$) are expressed power set, $\mathbb{U}$ is the subset. Let $\mathbb{A}$ be a subset that is contained with E. $\mathbb{A}$ can then be defined as a soft set by $\mathbb{U}$, which is denoted by the pair (ζ, $\mathbb{A}$), where ζ: $\mathbb{A} \rightarrow$ P($\mathbb{U}$) is a function mapping elements of $\mathbb{A}$ to subsets of the universal set $\mathbb{U}$. For any element e in A, the set $\zeta(e)$ can be interpreted as the set of approximate elements or elements within the soft set. Therefore, the soft set specified by (ζ, $\mathbb{A}$) is characterized by this mapping:

$$(\boldsymbol{\zeta}, \mathbb{A}) = \{\boldsymbol{\zeta}(\boldsymbol{e}) \in \mathbf{P}(\mathbb{U}) : \boldsymbol{e} \in \mathbf{E}, \boldsymbol{\zeta}(\boldsymbol{e}) = \emptyset \text{ } \mathbf{if}\, \boldsymbol{e} \neq \mathbb{A}\}. \tag{1}$$

2.2. Fuzzy Soft Sets

Consider $\mathbb{U}$ as the universal set and ∈ as the attribute set, with P($\mathbb{U}$) representing the power set of $\mathbb{U}$. If we assume that $\mathbb{A}$ is a subset of ∈, then the pair (ζ, $\mathbb{A}$) defines a fuzzy soft set. This is characterized by its mapping as follows:

$$\boldsymbol{\zeta} : \mathbb{A} \rightarrow \mathbf{P}(\mathbb{U}). \tag{2}$$

2.3. Hypersoft Sets

Let $\mathbb{P}(\mu)$ be the power set of μ. Considering $y^1, y^2, y^3, \dots, y^n$ when $n \geq 1$ and suppose n be well-defined attributes whose corresponding attributive elements are the set $\mathbb{Y}^1, \mathbb{Y}^2, \mathbb{Y}^3 \dots \mathbb{Y}^n$ with $\mathbb{Y}^i \cap \mathbb{Y}^j = \emptyset$, where $i \neq j$ and $i,j \epsilon \{1,2,3 \dots n\}$. Then, $(\xi, \mathbb{Y}^1 \times \mathbb{Y}^2 \times \mathbb{Y}^3 \dots \mathbb{Y}^n)$ is called a hypersoft set by μ, when:

$$\boldsymbol{\xi}: \mathbb{Y}^{\mathbf{1}} \times \mathbb{Y}^{\mathbf{2}} \times \mathbb{Y}^{\mathbf{3}} \dots \mathbb{Y}^{\boldsymbol{n}} \rightarrow \mathbb{P}(\boldsymbol{\mu}). \tag{3}$$

In Eq. (3), if we assign values to the attributes in the form of fuzzy (membership) only, then it is said to be a fuzzy hypersoft set.

3. Fuzzy Hypersoft Matrix

In this section, we present the definition, operations, laws along with theorem, and proposition on FHSM.

Let P be a set of parameters and $Let\ \mathbb{Y} = \{y_1, y_2, y_3, \dots y_\alpha\}$ be a finite set. The power set of $\mathbb{Y}$ is denoted by $(\mathbb{Y})$. Let $v^1, v^2, v^3 \dots v^n$ for $n \geq 1$ be n well-defined features, whose corresponding feature values are the sets $\beta^1, \beta^2, \beta^3, \dots \beta^n$ with $\beta^l \cap \beta^m = \emptyset$ for $l \neq m, l, m = 1,2 \dots n,$ respectively, and let their relation be $\upsilon = \beta^1 \times \beta^2 \times \beta^3 \times \dots \times \beta^n$. Then the pair (Ψ, υ) is called an FHSS over $\mathbb{Y}$, where $\Psi: \beta^1 \times \beta^2 \times \beta^3 \times \dots \times \beta^n \rightarrow P(\mathbb{Y})$ and $\Psi(\beta^1 \times \beta^2 \times \beta^3 \times \dots \times \beta^t) = \Psi(\upsilon)$, where $t \leq n$:

$$\boldsymbol{\mathcal{S}} = \{\langle \boldsymbol{\upsilon}, \boldsymbol{\mathcal{T}}(\boldsymbol{\Psi}(\boldsymbol{\upsilon})) \ \mathbf{y} \in \mathbb{Y} \rangle\}. \tag{4}$$

Let $\upsilon = \beta^1 \times \beta^2 \times \beta^3 \times \dots \times \beta^n$ be the relation, and its characteristic function is $\mathcal{X}_\upsilon: (\beta^1 \times \beta^2 \times \beta^3 \times \dots \times \beta^n) \rightarrow P(\mathbb{Y})$. It is defined as:

$$\boldsymbol{\mathcal{X}_\upsilon} = \left\{ \begin{matrix} \langle \boldsymbol{\upsilon}, \boldsymbol{\mathcal{T}}(\boldsymbol{\Psi}(\boldsymbol{\upsilon})) \ \mathbf{y} \in \mathbb{Y} \rangle, \\ \boldsymbol{\upsilon} \in (\boldsymbol{\beta}^{\mathbf{1}} \times \boldsymbol{\beta}^{\mathbf{2}} \times \boldsymbol{\beta}^{\mathbf{3}} \times \dots \times \boldsymbol{\beta}^{\boldsymbol{n}}) \end{matrix} \right\}. \tag{5}$$

If $\mathcal{S}_{ij} = \mathcal{X}_\upsilon(y^i, \beta_j^k)$, where $i = 1,2, \dots, \alpha$, $j = 1,2, \dots, \beta, k = 1,2, \dots, n,$ then a matrix is defined as:

$$[\boldsymbol{\mathcal{S}_{ij}}]_{\boldsymbol{\alpha \times \beta}} = \begin{pmatrix} \boldsymbol{\mathcal{S}_{11}} & \boldsymbol{\mathcal{S}_{12}} & \cdots & \boldsymbol{\mathcal{S}_{1\beta}} \\ \boldsymbol{\mathcal{S}_{21}} & \boldsymbol{\mathcal{S}_{22}} & \cdots & \boldsymbol{\mathcal{S}_{2\beta}} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\mathcal{S}_{\alpha 1}} & \boldsymbol{\mathcal{S}_{\alpha 2}} & \cdots & \boldsymbol{\mathcal{S}_{\alpha\beta}} \end{pmatrix}. \tag{6}$$

3.1. Transpose of Square Fuzzy Hypersoft Matrix

Let $\mathcal{S} = [\mathcal{S}_{ij}]$ be FHSM of order $\alpha \times \gamma$, where $\mathcal{S}_{ij} = (\boldsymbol{\mathcal{T}}^{\mathcal{S}}_{i\,jk})$. Then $\mathcal{S}^t$ is said to be the transpose of square FHSM if rows and columns of E are Interchange. It is denoted as.

$$\boldsymbol{\mathcal{S}^t} = [\boldsymbol{\mathcal{S}_{i\,j}}]^{\boldsymbol{t}} = [\boldsymbol{\mathcal{T}}^{\boldsymbol{\mathcal{S}}}_{\boldsymbol{i\,j\,k}}]^{\boldsymbol{t}} = [\boldsymbol{\mathcal{T}}^{\boldsymbol{\mathcal{S}}}_{\boldsymbol{j\,k\,i}}] = [\boldsymbol{\mathcal{T}_{j\,i}}]. \tag{7}$$

Proposition 3.5. Let $\mathcal{T} = [\mathcal{T}_{ij}]$ and $\mathcal{U} = [\mathcal{U}_{ij}]$ be two FHSMs, where $\mathcal{T}_{ij} = (\mathcal{T}^{\mathcal{T}}_{ijk})$ and $\mathcal{U}_{ij} = (\mathcal{T}^{\mathcal{U}}_{ijk})$. For two scalars $p, t \in [0,1]$, then:

i. $p(tT) = (pt)T$.
ii. If $p < t$, then $pT < tT$.
iii. If $T \subseteq U$, then $pT \subseteq pU$.

Proof of Proposition 3.5.

i. $p(tT) = s[tT_{ij}] = p[(tT_{ijk}^{T})] = [(ptT_{ijk}^{T})] = pt[(T_{ijk}^{T})] = pt[T^{T}{}_{ij}] = (pt)T$.
ii. Since $T_{ijk}^{T} \in [0,1], so$ $pT_{ijk}^{T} \leq tT_{ijk}^{T}$. Now $pT = [pT_{ij}] = [(pT_{ijk}^{T})] \leq [(tT_{ijk}^{T})] = [tT_{ij}] = tT$.
iii. $T \subseteq U \Rightarrow [T_{ij}] \subseteq [U_{ij}] \Rightarrow T_{ijk}^{T} \leq T_{ijk}^{U} \Rightarrow pT_{ijk}^{T} \leq pT_{ijk}^{U} \Rightarrow p[T_{ij}] \subseteq p[U_{ij}] \Rightarrow pT \subseteq pU$.

Theorem 3.6. Let $T = [T_{ij}]$ be the FHSM of order $\alpha \times \gamma$, where $T_{ij} = (T_{ijk}^{T})$. Then:

i. $(pT)^t = pT^t, where\ p \in [0,1]$.
ii. $(T^t)^t = T$.

Proof of Theorem 3.6:

i. Here $(pT)^t, pT^t \in FHSM_{\alpha\times\gamma}$, so $(pT)^t = [(pT_{ijk}^{T})]^t = [(pT_{jki}^{T})] = p[(T_{jki}^{T})] = p[(T_{ijk}^{T})]^t = pT^t$.
ii. Since $T^t \in FHSM_{\alpha\times\gamma}\ To\ (T^t)^t \in FHSM_{\alpha\times\gamma}$. Now, $(T^t)^t = \left([(T_{ijk}^{T})]^t\right)^t = \left([(T_{jki}^{T})]\right)^t = [(T_{ijk}^{T})] = T$.

3.2. Trace of Fuzzy Hypersoft Matrix

Let $T = [T_{ij}]$ be the square FHSM of order $\alpha \times \gamma$, where $T_{ij} = (T_{ijk}^{T})$, and $\alpha = \gamma$. Then, a trace of FHSM is denoted as $tr(T)$ and is defined as:

$$\boldsymbol{tr(T) = \sum_{i=1,k=a}^{\alpha,z}[T_{iik}^{T}]}. \tag{8}$$

Proposition 3.9. Let $T = [T_{i\,j}]$ be the square FHSM of order $\alpha \times \gamma$, where $T_{i\,j} = (T_{i\,j\,k}^{T})$ and $\alpha \times \gamma$. P be any scalar, then $tr(pT) = p\ tr(T)$.

Proof of Proposition 3.9. $tr(sO) = \sum_{i=1,k=a}^{\alpha,z}[pT_{iik}^{T}] = p\sum_{i=1,k=a}^{\alpha,z}[T_{iik}^{T}] = p\ tr(T)$.

3.3. Max-Min Product of Fuzzy Hypersoft Matrix

Let $T = [T_{ij}]$ and $U = [U_{jm}]$ be two FHSMs, where $T_{ij} = (T_{ijk}^{T})$ and $U_{jm} = (T_{jkm}^{T})$. Then, if the dimensions of S and U are equal (the number of columns in S equals the number of rows in A), they are considered conformable. If $T = [T_{ij}]_{\alpha\times\beta}$ and $U = [U_{jm}]_{\beta\times\gamma}$, then $T \otimes U = [S_{im}]_{\alpha\times\gamma}$.

Theorem 3.11. Let $= [\mathcal{T}_{ij}]$, $\mathcal{U} = [\mathcal{U}_{ij}]$ and $\mathcal{R} = [\mathcal{R}_{ij}]$ be FHSM where $\mathcal{T}_{ij} = (\mathcal{T}^{\mathcal{T}}_{ijk})$, $\mathcal{U}_{ij} = (\mathcal{T}^{\mathcal{U}}_{ijk})$ and $\mathcal{R}_{ij} = (\mathcal{T}^{\mathcal{R}}_{ijk})$. Then:

i. $\mathcal{T} \cap (\mathcal{U} \oplus \mathcal{R}) = (\mathcal{T} \cap \mathcal{U}) \oplus (\mathcal{T} \cap \mathcal{R})$.
ii. $(\mathcal{T} \oplus \mathcal{U}) \cap \mathcal{R} = (\mathcal{T} \cap \mathcal{R}) \oplus (\mathcal{U} \cap \mathcal{R})$.
iii. $\mathcal{T} \cup (\mathcal{U} \oplus \mathcal{R}) = (\mathcal{T} \cup \mathcal{U}) \oplus (\mathcal{T} \cup \mathcal{R})$.
iv. $(\mathcal{T} \oplus \mathcal{U}) \cup \mathcal{R} = (\mathcal{T} \cup \mathcal{R}) \oplus (\mathcal{U} \cup \mathcal{R})$.

Proof of Theorem 3.11.

i. $\mathcal{T} \cap (\mathcal{U} \oplus \mathcal{R}) = (\mathcal{T}^{\mathcal{T}}_{ijk}) \cap \left[\left(\frac{(\mathcal{T}^{\mathcal{T}}_{ijk} + \mathcal{T}^{\mathcal{R}}_{ijk})}{2}\right)\right] = \left[\left(\min\left(\mathcal{T}^{\mathcal{T}}_{ijk}, \frac{(\mathcal{T}^{\mathcal{U}}_{ijk} + \mathcal{T}^{\mathcal{R}}_{ijk})}{2}\right)\right)\right] = \left[\left(\min\left(\frac{(\mathcal{T}^{\mathcal{T}}_{ijk} + \mathcal{T}^{\mathcal{U}}_{ijk})}{2}, \frac{(\mathcal{T}^{\mathcal{T}}_{ijk} + \mathcal{T}^{\mathcal{R}}_{ijk})}{2}\right)\right)\right] = [(\min(\mathcal{T}^{\mathcal{T}}_{ijk}, \mathcal{T}^{\mathcal{U}}_{ijk}))] \oplus [(\min(\mathcal{T}^{\mathcal{T}}_{ijk}, \mathcal{T}^{\mathcal{R}}_{ijk}))] = [(\mathcal{T}^{\mathcal{T}}_{ijk}) \cap (\mathcal{T}^{\mathcal{U}}_{ijk})] \oplus [(\mathcal{T}^{\mathcal{T}}_{ijk}) \cap (\mathcal{T}^{\mathcal{R}}_{ijk})] = (\mathcal{T} \cap \mathcal{U}) \oplus (\mathcal{T} \cap \mathcal{R})$.

ii. $(\mathcal{T} \oplus \mathcal{U}) \cap \mathcal{R} = \left[\left(\frac{(\mathcal{T}^{\mathcal{T}}_{ijk} + \mathcal{T}^{\mathcal{U}}_{ijk})}{2}\right)\right] \cap (\mathcal{T}^{\mathcal{R}}_{ijk}) = \left[\left(\min\left(\frac{(\mathcal{T}^{\mathcal{T}}_{ijk} + \mathcal{T}^{\mathcal{U}}_{ijk})}{2}, \mathcal{T}^{\mathcal{R}}_{ijk}\right)\right)\right] = \left[\left(\min\left(\frac{(\mathcal{T}^{\mathcal{T}}_{ijk} + \mathcal{T}^{\mathcal{R}}_{ijk})}{2}, \frac{(\mathcal{T}^{\mathcal{U}}_{ijk} + \mathcal{T}^{\mathcal{R}}_{ijk})}{2}\right)\right)\right] = [(\min(\mathcal{T}^{\mathcal{T}}_{ijk}, \mathcal{T}^{\mathcal{R}}_{ijk}))] \oplus [(\min(\mathcal{T}^{\mathcal{U}}_{ijk}, \mathcal{T}^{\mathcal{R}}_{ijk}))] = [(\mathcal{T}^{\mathcal{T}}_{ijk}) \cap (\mathcal{T}^{\mathcal{R}}_{ijk})] \oplus [(\mathcal{T}^{\mathcal{U}}_{ijk}) \cap (\mathcal{T}^{\mathcal{R}}_{ijk})] = (\mathcal{T} \cap \mathcal{R}) \oplus (\mathcal{U} \cap \mathcal{R})$.

iii. $\mathcal{T} \cup (\mathcal{U} \oplus \mathcal{R}) = (\mathcal{T}^{\mathcal{T}}_{ijk}) \cup \left[\left(\frac{(\mathcal{T}^{\mathcal{T}}_{ijk} + \mathcal{T}^{\mathcal{R}}_{ijk})}{2}\right)\right] = \left[\left(\max\left(\mathcal{T}^{\mathcal{T}}_{ijk}, \frac{(\mathcal{T}^{\mathcal{U}}_{ijk} + \mathcal{T}^{\mathcal{R}}_{ijk})}{2}\right)\right)\right] = \left[\left(\max\left(\frac{(\mathcal{T}^{\mathcal{T}}_{ijk} + \mathcal{T}^{\mathcal{U}}_{ijk})}{2}, \frac{(\mathcal{T}^{\mathcal{T}}_{ijk} + \mathcal{T}^{\mathcal{R}}_{ijk})}{2}\right)\right)\right] = [(\max(\mathcal{T}^{\mathcal{T}}_{ijk}, \mathcal{T}^{\mathcal{U}}_{ijk}))] \oplus [(\max(\mathcal{T}^{\mathcal{T}}_{ijk}, \mathcal{T}^{\mathcal{R}}_{ijk}))] = [(\mathcal{T}^{\mathcal{T}}_{ijk}) \cup (\mathcal{T}^{\mathcal{U}}_{ijk})] \oplus [(\mathcal{T}^{\mathcal{T}}_{ijk}) \cup (\mathcal{T}^{\mathcal{R}}_{ijk})] = (\mathcal{T} \cup \mathcal{U}) \oplus (\mathcal{T} \cup \mathcal{R})$.

iv. $(\boldsymbol{\mathcal{T}} \oplus \mathcal{U}) \cup \mathcal{R} = \left[\left(\frac{(\boldsymbol{\mathcal{T}}^{\mathcal{T}}_{ijk} + \boldsymbol{\mathcal{T}}^{\mathcal{U}}_{ijk})}{2}\right)\right] \cup (\boldsymbol{\mathcal{T}}^{\mathcal{R}}_{ijk}) = \left[\left(\max\left(\frac{(\boldsymbol{\mathcal{T}}^{\mathcal{T}}_{ijk} + \boldsymbol{\mathcal{T}}^{\mathcal{U}}_{ijk})}{2}, \boldsymbol{\mathcal{T}}^{\mathcal{R}}_{ijk}\right)\right)\right] = \left[\left(\max\left(\frac{(\boldsymbol{\mathcal{T}}^{\mathcal{T}}_{ijk} + \boldsymbol{\mathcal{T}}^{\mathcal{R}}_{ijk})}{2}, \frac{(\boldsymbol{\mathcal{T}}^{\mathcal{U}}_{ijk} + \boldsymbol{\mathcal{T}}^{\mathcal{R}}_{ijk})}{2}\right)\right)\right] = [(\max(\boldsymbol{\mathcal{T}}^{\mathcal{T}}_{ijk}, \boldsymbol{\mathcal{T}}^{\mathcal{R}}_{ijk}))] \oplus [(\max(\boldsymbol{\mathcal{T}}^{\mathcal{U}}_{ijk}, \boldsymbol{\mathcal{T}}^{\mathcal{R}}_{ijk}))] = [(\boldsymbol{\mathcal{T}}^{\mathcal{T}}_{ijk}) \cup (\boldsymbol{\mathcal{T}}^{\mathcal{R}}_{ijk})] \oplus [(\boldsymbol{\mathcal{T}}^{\mathcal{U}}_{ijk}) \cup (\boldsymbol{\mathcal{T}}^{\mathcal{R}}_{ijk})] = (\boldsymbol{\mathcal{T}} \cup \mathcal{R}) \oplus (\mathcal{U} \cup \mathcal{R})$.

4. Algorithm

For decision-making in uncertain contexts, MCDM algorithms are invaluable tools, especially when applied to fuzzy hypersoft matrices. These algorithms handle alternatives and uncertainties present in real-world scenarios by considering several factors and using fuzzy logic to enable more robust and informed decision-making. Decision-makers obtain an understanding of their practical utility in a variety of sectors, including engineering, finance, and healthcare, through case studies that illustrate their applicability. The steps of our algorithm are:

Step 1 – Create an FHSM using Eq. (6).

Step 2 – Create a value matrix for FHSM. Let $\mathcal{T} = [\mathcal{T}_{ij}]$ be the order's FHSM, where $\mathcal{T}_{ij} = (\mathcal{T}^{\mathcal{T}}_{ijk})$. The value of matrix *S* is therefore denoted by V(*S*) with (S)= $[\mathcal{V}^{\mathcal{T}}_{ij}]$ of order $\alpha \times \gamma$, where$\mathcal{V}^{\mathcal{T}}_{ij} = \mathcal{T}^{\mathcal{T}}_{ijk}$.

Step 3 – Value matrices are used to calculate the score matrix. The score of two FHSM $\mathcal{T} = [\mathcal{T}_{ij}]$ and $\mathcal{U} = [\mathcal{U}_{ij}]$ of order $\alpha \times \gamma$ is given as $\mathcal{S}(\mathcal{T},\mathcal{U}) = \mathcal{V}(\mathcal{T}) + \mathcal{V}(\mathcal{U})$ and $\mathcal{S}(\mathcal{T},\mathcal{U}) = [\mathcal{S}_{ij}]$, where $\mathcal{S}_{ij} = \mathcal{V}_{ij}^{\mathcal{T}} + \mathcal{V}_{ij}^{\mathcal{U}}$.

Step 4 – Utilize the score matrix to determine the overall score. The total rating of each item in the universal set is $\left|\sum_{j=1}^{n} \mathcal{S}_{ij}\right|$.

Step 5 – To discover the best answer, choose the item with the highest score from the total score matrix.

Figure 1 presents the proposed algorithm.

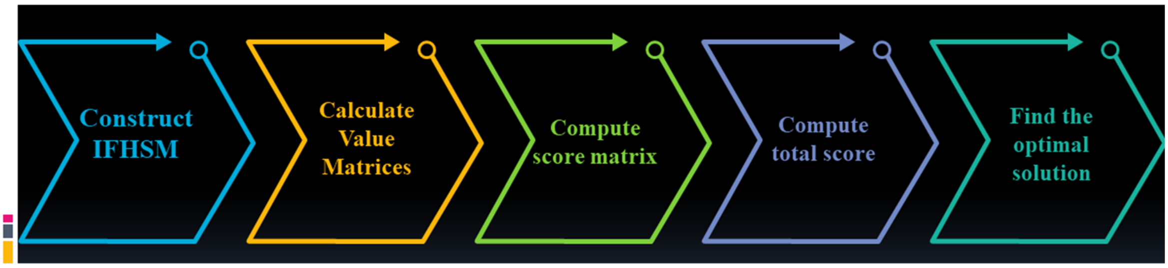

Fig. 1. Proposed Algorithm.

5. Case Study

Let $\mathbb{Y} = \{\delta^1, \delta^2, \delta^3, \delta^4, \delta^5\}$ be the set of options. Attributes are classified as "performance" (β^1) (i.e. to select a performance according to scale={1, 2, 3, 4}), "grades" (β^2) (with grades as {A, A-, B+, B}), "research" (β^3) (i.e. research score according to educational department as {AAA, BBB, CCC}), and "institute" (β^4) (i.e. government, private, and semi-government). The mapping is defined as $\Psi: (\beta^1 \times \beta^2 \times \beta^3 \times \beta^4) \to P(\mathbb{Y})$. Let us assume:

$\mathcal{S} = \boldsymbol{\Psi}\,(3, \text{A}-, \text{CCC}, \text{gov}) = \{\langle \delta^1, (3\ (0.3), A-\ (0.4), CCC\ (0.6), gov\ (0.4)\rangle, \langle \delta^3, (3\ (0.2), A-\ (0.5), CCC\ (0.8), gov(0.5)\rangle, \langle \delta^9, (3\ (0.3), A-\ (0.4), CCC\ (0.5), gov\ (0.2)\rangle, \langle \delta^{15}, (3\ (0.4), A-\ (0.6), CCC\ (0.2), gov(0.5)\rangle\}$. Also:

$\mathcal{T} = \boldsymbol{\Psi}\,(3, \text{A}-, \text{CCC}, \text{gov}) = \{\langle \delta^1, (3\ (0.1), A-\ (0.4), CCC\ (0.4), gov\ (0.3)\rangle, \langle \delta^3, (3\ (0.2), A-\ (0.6), CCC\ (0.9), gov(0.6)\rangle, \langle \delta^9, (3(0.2), A-\ (0.7), CCC\ (0.6), gov\ (0.5)\rangle, \langle \delta^{15}, (3\ (0.6), A-\ (0.4), CCC\ (0.4), gov(0.4)\rangle\}$.

Then, we apply the algorithm for the calculation of total values:

Step 1 – The above two sets of FHSSs are given as FHSMs:

$$[\mathcal{S}] = \begin{bmatrix} 3(0.3) & A-(0.4) & CCC(0.6) & gov(0.4) \\ 3(0.2) & A-(0.5) & CCC(0.8) & 43g(0.5) \\ 3(0.3) & A-(0.4) & CCC(0.5) & gov(0.2) \\ 3(0.4) & A-(0.6) & CCC(0.2) & gov(0.5) \end{bmatrix},$$

$$[\mathcal{T}] = \begin{bmatrix} 3(0.1) & A-(0.4) & CCC(0.4) & gov(0.3) \\ 3(0.2) & A-(0.6) & CCC(0.9) & gov(0.6) \\ 3(0.2) & A-(0.7) & CCC(0.6) & gov(0.5) \\ 3(0.6) & A-(0.4) & CCC(0.4) & gov(0.4) \end{bmatrix}.$$

Step 2 – Now calculate the values matrices of FHSMs as:

$$[v((\mathcal{S})] = \begin{bmatrix} 3(0.3) & A-(0.4) & CCC(0.6) & gov(0.4) \\ 3(0.2) & A-(0.5) & CCC(0.8) & gov(0.5) \\ 3(0.3) & A-(0.4) & CCC(0.5) & gov(0.2) \\ 3(0.4) & A-(0.6) & CCC(0.2) & gov(0.5) \end{bmatrix},$$

$$[v(\mathcal{T})] = \begin{bmatrix} 3(0.1) & A-(0.4) & CCC(0.4) & gov(0.3) \\ 3(0.2) & A-(0.6) & CCC(0.9) & gov(0.6) \\ 3(0.2) & A-(0.7) & CCC(0.6) & gov(0.5) \\ 3(0.6) & A-(0.4) & CCC(0.4) & gov(0.4) \end{bmatrix}.$$

Step 3 – Compute the score matrix by combining the value matrices as:

$$(\mathcal{S}(\mathcal{S},\boldsymbol{\mathcal{T}}) = \begin{bmatrix} 3(0.4) & A-(0.8) & CCC(1.0) & gov(0.7) \\ 3(0.4) & A-(1.1) & CCC(1.7) & gov(1.1) \\ 3(0.5) & A-(1.1) & CCC(1.1) & gov(0.7) \\ 3(1.0) & A-(1.0) & CCC(0.6) & gov(0.9) \end{bmatrix}.$$

Step 4 – Compute a total score as:

$$Total\ Score = \begin{bmatrix} 1.2 \\ 2.3 \\ 4.1 \\ 1.9 \end{bmatrix}.$$

Step 5 – δ^3 will be the best choice.

6. Discussion and Comparison

Through the comparative assessment presented in Table 1. Furthermore, in the context of decision-making, our approach provides a richer informational basis for navigating the uncertainties inherent in data. Additionally, numerous configurations of FS composite structure are encapsulated as specific instances within FHSM. Our approach allows for a more precise and empirical representation of information pertaining to the subject matter, making it an advantageous tool for integrating imprecise and uncertain data within decision-making frameworks. Consequently, our method demonstrates effectiveness, adaptability, simplicity, and superiority.

Table 1
The result comparison with existing studies

	Set	Truthiness	Attributive	Sub-attributive	Parametrization	Advantages
Zadeh [1]	**FS**	✓	×	✓	×	**Addresses uncertainty through the application of fuzzy intervals**
Maji et al. [9]	**FSM**	✓	×	✓	✓	**Addresses uncertainty through the application of fuzzy soft intervals**
Proposed	**FHSM**	✓	✓	✓	✓	**Addresses uncertainty through the application of FHSM**

7. Conclusion

This research delves into the essential characteristics, aggregation processes, and foundational principles of fuzzy set theory, emphasizing their applications and relevance in the context of hypersoft sets and hypersoft matrices. It further explores the critical aspects and elementary operations of matrices within this unique set environment. Additionally, the study outlines prospective pathways for future research, highlighting the potential for creating novel hybrid models by combining hypersoft sets with other mathematical frameworks such as fuzzy sets, rough sets, expert sets, and cubic sets. It also proposes the exploration of advanced algebraic constructs, including the development of hypersoft topological spaces, functional spaces, groups, vector spaces, rings, and measures. These future directions aim to expand the utility and understanding of hypersoft sets, offering innovative approaches to complex problem-solving and theoretical advancement in the field.

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