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Study of the System of Uncertain Linear Differential Equations Under Neutrosophic Sense of Uncertainty

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ABSTRACT

This paper considers a system of uncertain linear differential equations under the Neutrosophic uncertain environment. Suppose the mutual dependency of the dynamics of two variables is set on the mathematical manipulation under the scenario where the available information regarding initial states is imprecise and is given in the Neutrosophic sense of uncertainty. The concept of Neutrosophic differential equations can play a very effective role in this regard. The present chapter is engaged with a brief introduction of the theory of the system of Neutrosophic differential equations and the possible solution approaches. Combining different cases of Neutrosophic differentiability and the signs of the coefficients involved in the differential equations are taken for the synergetic study of the theory. At the end of the theory, a few physical phenomena are explored as possible applications of the proposed theory. For a better understanding and reliability of the proposed theory, numerical simulations and graphical visualizations are added to different pockets of this paper.

1. Introduction

In the real world, very frequently, the available data is not always crisp and precise in nature. There must be some inherited ambiguity and uncertainty about the data, which causes real-world decision making and mensuration to be imprecise. One of the celebrated mathematical objects carrying the sense of uncertainty is the theory of fuzzy set and logic [1] introduced by Lotfi A. Zadeh in 1965. Later, more developments in mathematical structures were done by the works of some eminent researchers (see [2-4]). Later, the sense of fuzzy uncertainty was generalized by adding the notion of non-belongingness promoting the Intuitionistic fuzzy set by Atanassov [5]. Smarandache [6] contributed a notion of more generalized type of uncertainty introducing Neutrosophic

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philosophy. The measure of the membership, non-membership and indeterminacy grades is done using Neutrosophic logic [7-9]. Upon the introduction of the theory, it gains major attention from researchers working on the quantification and qualitative simulation of physical problems in uncertain scenarios. In this scenario, a significant contribution on the establishment of the Neutrosophic theory was included by Wang et al., [10]. An implementation of the Neutrosophic logic on the shortest path problems are discussed by Kumar et al., [11]. Chakraborty et al., [12-13] supplied many pentagonal Neutrosophic number structures. Some novel definitions of the Neutrosophic sets (namely, Type 2 NS, Bipolar NS, Cylindrical NS, Spherical NS, Pythagorean NS, Cauchy NS) were established in the recent works on this domain [14-18].

The theories of differential equations were utilized to capture the dynamical behavior of the physical scenarios. In this context, the theory of fuzzy differential equations is rapidly grown as the fundamental tool to deal with dynamical situations under uncertainties. The notion of the whereas the fuzzy differential equation was first established by Keleva [19]. The concept of generalized Hukuhara derivative was developed and utilized to demonstrate the fuzzy differential equations [20-23]. Allahviranloo and Ahmadi [24] solved linear fuzzy differential equations using the Laplace transformation approach. Ghanbari [25] solved the linear FDE by the Lagrange multiplier method using the Hukuhara derivative. Recently, Rahaman et al., [26] contributed literature exploring the Gaussian fuzzy number and its utilization on the solution of fuzzy difference equations. Also, the fractional differential equations were utilized by Rahaman et al., [27] to describe the memory sensitive study of inventory control problems in a crisp and fuzzy environment. Manna et al., [28] and Garg et al., [29] solve the inventory model in the fuzzy environment on the applications on carbon emitted production industry and scrap and defective items inspection in an IoT-size, respectively. Further, Pattnaik et al., [30] solve the control inventory model for the COVID-19 framework using Neutrosophic uncertainty. Mondal and Roy [31] discussed the solution of a system of differential equations with intuitionistic fuzzy number valued initial states. Smarandache [32] was the pioneer to introduce the Neutrosophic derivative as an extension of the fuzzy derivative. A novel type of Neutrosophic derivative, namely the granular derivative was formulated by Son et al., [33]. Very recently, some researchers have initiated research works on the neutrosophic differential equations. In this context, Sumanthi et al., [34] suggested a technique for resolving differential equations involving Neutrosophic numbers and recommended an application of the proposed theory to the bacteria culture model. Later, Sumanthi et al., [35] explored the solutions of the Neutrosophic differential equation taking trapezoidal Neutrosophic numbers as the boundary conditions. Another recent work on the second order boundary value problem by Moi et al., [36] in the Neutrosophic arena is spotted in our literature survey.

In this present paper, a system of linear homogeneous differential equations is interpreted in a Neutrosophic environment considering the generalized Neutrosophic derivative of the dependent variables and Neutrosophic initial states. An application of the proposed theory on the arms race model is discussed, taking the initial information in terms of the newly introduced Cauchy Neutrosophic numbers.

The rest of the article is structured as follows: The preliminary theory regarding the Neutrosophic number and Neutrosophic derivative is given in Section 2. In Section 3, the main theory for the solution of the system of Neutrosophic differential equations is established. Further, the application of the proposed theory is hinted in Section 4. Lastly, the conclusion over the whole paper is made in Section 5.

2. Preliminaries

Preliminaries of mathematical tools are briefly discussed in this section. The concept of the Neutrosophic set [6] was first introduced by Professor Florentin Smarandache in 1998. Here, every element of the set has three membership functions to describe the belongingness of the element in the set more specifically than the fuzzy set [1]. Details discussion on the Neutrosophic set are as follows:

Definition 1: [6] Consider Neutrosophic set $\widetilde{A_N}$ over the universal set of discourse Y is defined by $\widetilde{A_N} = \{< x, T_{\widetilde{A_N}}(x), I_{\widetilde{A_N}}(x), F_{\widetilde{A_N}}(x) >: x \in Y\}$, where $T_{\widetilde{A_N}}(x), I_{\widetilde{A_N}}(x), F_{\widetilde{A_N}}(x): Y \to (0,1)$ are three functions satisfying $0^- \leq T_{\widetilde{A_N}}(x) + I_{\widetilde{A_N}}(x) + F_{\widetilde{A_N}}(x) \leq 3^+$ for $x \in Y$.

Here, $T_{\widetilde{A_N}}(x), I_{\widetilde{A_N}}(x), F_{\widetilde{A_N}}(x)$ are called the membership degree of truthiness, indeterminacy and falsity for all $x \in Y$, respectively.

Definition 2: [13] Assume a neutrosophic set $\widetilde{A_N}$ over the universal set of discourse Y is called to be single valued if x is a single valued independent variable. Then, $T_{\widetilde{A_N}}(x), I_{\widetilde{A_N}}(x), F_{\widetilde{A_N}}(x): X \to [0,1]$ will present the truth, indeterminacy and falsity membership functions, respectively.

Definition 3: [6] The (α, β, γ) −cut of neutrosophic set $\widetilde{A_N}$ is defined by $\widetilde{A_{N(\alpha,\beta,\gamma)}}$, where $\alpha, \beta, \gamma \in [0,1]$ are fixed numbers such that $\alpha + \beta + \gamma \leq 3$ and is defined by

$$\widetilde{A_{N(\alpha,\beta,\gamma)}} = \{x \in X: T_{\widetilde{A_N}}(x) \geq \alpha, I_{\widetilde{A_N}}(x) \leq \beta, F_{\widetilde{A_N}}(x) \leq \gamma\} \tag{1}$$

Definition 4: [13] A Neutrosophic set $\widetilde{A_N}$ is said to be a neutrosophic number over a set of real numbers ($\mathbb{R}$) if it has the following properties:

(i) $\widetilde{A_N}$ is neutro-normal if there exists $x_0 \in \mathbb{R}$ such that $T_{\widetilde{A_N}}(x_0) = 1$ and $I_{\widetilde{A_N}}(x_0) = 0 = F_{\widetilde{A_N}}(x_0)$.

(ii) $\widetilde{A_N}$ is a convex set for the truth function $(T_{\widetilde{A_N}})$; i.e., $T_{\widetilde{A_N}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{T_{\widetilde{A_N}}(x_1), T_{\widetilde{A_N}}(x_2)\}$, for $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0,1]$.

(iii) $\widetilde{A_N}$ is a concave set for the indeterminacy function $(I_{\widetilde{A_N}})$; i.e., $I_{\widetilde{A_N}}(\lambda x_1 + (1-\lambda)x_2) \leq \max\{I_{\widetilde{A_N}}(x_1), I_{\widetilde{A_N}}(x_2)\}$, for $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0,1]$.

(iv) $\widetilde{A_N}$ is a concave set for the falsity function $(F_{\widetilde{A_N}})$; i.e., $F_{\widetilde{A_N}}(\lambda x_1 + (1-\lambda)x_2) \leq \max\{F_{\widetilde{A_N}}(x_1), F_{\widetilde{A_N}}(x_2)\}$, for $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0,1]$.

If the conditions (ii)-(iv) are satisfied, then the Neutrosophic set $(\widetilde{A_N})$ is called neutro-convex.

Definition 5: [13] If $\widetilde{A_N}$ is a Neutrosophic number, then (α, β, γ)-cut is given by $\widetilde{A_{N(\alpha,\beta,\gamma)}} = \{[A_1(\alpha), A_2(\alpha)], [A_1'(\beta), A_2'(\beta)], [A_1''(\gamma), A_2''(\gamma)]\}$, where

(i) $\frac{dA_1(\alpha)}{d\alpha} > 0, \frac{dA_2(\alpha)}{d\alpha} < 0, \ \forall\, \alpha \in [0,1], \ A_1(1) \leq A_2(1)$

(ii) $\frac{dA_1'(\beta)}{d\beta} < 0, \frac{dA_2'(\beta)}{d\beta} > 0, \ \forall\, \beta \in [0,1], \ A_1'(0) \leq A_2'(0)$

(iii) $\frac{dA_1''(\gamma)}{d\gamma} < 0, \frac{dA_2''(\gamma)}{d\gamma} > 0, \forall\, \gamma \in [0,1], \ A_1''(0) \leq A_2''(0)$

with $\alpha + \beta + \gamma \leq 3$.

Definition 6: [18] A symmetrical Bell-shaped Neutrosophic number (BNN) is denoted by $\widetilde{B_N}$ and describe as

$$\widetilde{B_N} =< (u_1; v_1; w), (u_2; v_2; w), (u_3; v_3; w) > \tag{2}$$

where the truth, indeterminacy and falsity membership functions are denoted by $T_{\widetilde{B_N}}(x)$, $I_{\widetilde{B_N}}(x)$ and $F_{\widetilde{B_N}}(x)$, respectively and defined as the following

$$\begin{cases} T_{\widetilde{B_N}}(x) = \frac{1}{1+\left|\frac{x-w}{u_1}\right|^{2v_1}} \\ I_{\widetilde{B_N}}(x) = 1 - \frac{1}{1+\left|\frac{x-w}{u_2}\right|^{2v_2}} \\ F_{\widetilde{B_N}}(x) = 1 - \frac{1}{1+\left|\frac{x-w}{u_3}\right|^{2v_3}} \end{cases} \tag{3}$$

where $-\infty < x < \infty$ and $u_i, v_i, w\ (i = 1,2,3)$ are the three variables. Normally, v_i be the parameters are positive for $i = 1,2,3$. Further, w be another parameter locates at the centre of the curve and v_i be the parameters control the slopes at the truth, indeterminacy and falsity membership functions crossover points, respectively, for $i = 1,2,3$.

If we consider the value of $v_i = 1$ for all $i = 1,2,3$, then the above definition of the bell-shaped Neutrosophic number is used to get the following definition.

Definition 7: [18] A symmetrical Cauchy neutrosophic number (CNN) is denoted by

$$\widetilde{C_N} =< (u_1; w), (u_2; w), (u_3; w) > \tag{4}$$

is described by the truth, indeterminacy and falsity membership functions $T_{\widetilde{C_N}}(x)$, $I_{\widetilde{C_N}}(x)$ and $F_{\widetilde{C_N}}(x)$, respectively and defined as the following:

$$\begin{cases} T_{\widetilde{C_N}}(x) = \frac{1}{1+\left(\frac{x-w}{u_1}\right)^2} \\ I_{\widetilde{C_N}}(x) = 1 - \frac{1}{1+\left(\frac{x-w}{u_2}\right)^2} \\ F_{\widetilde{C_N}}(x) = 1 - \frac{1}{1+\left(\frac{x-w}{u_3}\right)^2} \end{cases} \tag{5}$$

where $x \in (-\infty, \infty)$.

Definition 8: [18] The parametric form of symmetrical CNN is described by

$$\widetilde{C_N} =< (u_1; w), (u_2; w), (u_3; w) > \tag{6}$$

is given by $\widetilde{C_N}_{\alpha\in(0,1],\beta,\gamma\in[0,1)} = \left[\left(C_{T_l}(\alpha), C_{T_r}(\alpha)\right); \left(C_{I_l}(\beta), C_{I_r}(\beta)\right); \left(C_{F_l}(\gamma), C_{F_r}(\gamma)\right)\right]$, where

$$\begin{cases} C_{T_l}(\alpha) = w - u_1\sqrt{\frac{1-\alpha}{\alpha}} \\ C_{T_r}(\alpha) = w + u_1\sqrt{\frac{1-\alpha}{\alpha}} \\ C_{I_l}(\beta) = w - u_2\sqrt{\frac{\beta}{1-\beta}} \\ C_{I_r}(\beta) = w + u_2\sqrt{\frac{\beta}{1-\beta}} \\ C_{F_l}(\gamma) = w - u_3\sqrt{\frac{\gamma}{1-\gamma}} \\ C_{F_r}(\gamma) = w + u_3\sqrt{\frac{\gamma}{1-\gamma}} \end{cases} \tag{7}$$

Definition 9: [36] Consider a Neutrosophic valued function (f) define as $f: I \to N$ and $x_0 \in I$. Then, the generalized Neutrosophic derivative of $f(x)$ at x_0 is denoted by $f'(x_0)$ and defined as follows

(i) $f'_{T_\alpha} = \left[\min\left\{f'_{T_L}(x_0;\alpha), f'_{T_R}(x_0;\alpha)\right\}, \max\left\{f'_{T_L}(x_0;\alpha), f'_{T_R}(x_0;\alpha)\right\}\right]$ if $f'_{T_L}(x_0;\alpha), f'_{T_R}(x_0;\alpha)$ exist.

(ii) $f'_{I_\beta} = \left[\min\left\{f'_{I_L}(x_0;\beta), f'_{I_R}(x_0;\beta)\right\}, \max\left\{f'_{I_L}(x_0;\beta), f'_{I_R}(x_0;\beta)\right\}\right]$ if $f'_{I_L}(x_0;\beta), f'_{I_R}(x_0;\beta)$ exist.

(iii) $f'_{F_\gamma} = \left[\min\left\{f'_{F_L}(x_0;\gamma), f'_{F_R}(x_0;\gamma)\right\}, \max\left\{f'_{F_L}(x_0;\gamma), f'_{F_R}(x_0;\gamma)\right\}\right]$ if $f'_{F_L}(x_0;\gamma), f'_{F_R}(x_0;\gamma)$ exist.

$f'(x)$ is said to be of type I derivative if

$$[f(x_0)]'_{(\alpha,\beta,\gamma)} = \left\{\left[f'_{T_L}(x_0;\alpha), f'_{T_R}(x_0;\alpha)\right], \left[f'_{I_L}(x_0;\beta), f'_{I_R}(x_0;\beta)\right], \left[f'_{F_L}(x_0;\gamma), f'_{F_R}(x_0;\gamma)\right]\right\} \tag{8}$$

And of type II derivative if

$$[f(x_0)]'_{(\alpha,\beta,\gamma)} = \left\{\left[f'_{T_R}(x_0;\alpha), f'_{T_L}(x_0;\alpha)\right], \left[f'_{I_R}(x_0;\beta), f'_{I_L}(x_0;\beta)\right], \left[f'_{F_R}(x_0;\gamma), f'_{F_L}(x_0;\gamma)\right]\right\} \tag{9}$$

3. System of linear homogenous Neutrosophic Differential Equations

Consider the system of the linear homogeneous differential equation of the form

$$\begin{cases} \frac{dy}{dx} = Az \\ \frac{dz}{dx} = By \end{cases} \tag{10}$$

In the Equation (10), A, B are constants. The initial conditions are given by

$$\begin{cases} y(x_0) = y_0 \\ z(x_0) = z_0 \end{cases} \tag{11}$$

The system represented by Equation (10) and Equation (11) is called a system of the homogenous linear Neutrosophic differential equation when the initial conditions given by Equation (11) are considered to be the Neutrosophic numbers.

Suppose the initial values are $y(x_0) = y_0$ and $z(x_0) = z_0$ of Neutrosophic numbers and are denoted by

$$\tilde{y}_{Neu}(0) = \left[\left(y_{0L}(\alpha), y_{0R}(\alpha)\right); \left(y'_{0L}(\beta), y'_{0R}(\beta)\right); \left(y''_{0L}(\gamma), y''_{0R}(\gamma)\right)\right] \tag{12}$$

and

$$\tilde{z}_{Neu}(0) = \left[\left(z_{0L}(\alpha), z_{0R}(\alpha)\right); \left(z'_{0L}(\beta), z'_{0R}(\beta)\right); \left(z''_{0L}(\gamma), z''_{0R}(\gamma)\right)\right] \tag{13}$$

Then Equations (10) and (11) can be represented as,

$$\begin{cases} \frac{d\tilde{y}_{Neu}}{dx} = A\,\tilde{z}_{Neu} \\ \frac{d\tilde{z}_{Neu}}{dx} = B\,\tilde{y}_{Neu} \end{cases} \tag{14}$$

with the initial condition

$$\begin{cases} \tilde{y}_{Neu}(x_0) = \tilde{y}_{0\,Neu} \\ \tilde{z}_{Neu}(x_0) = \tilde{z}_{0\,Neu} \end{cases} \tag{15}$$

Rewriting the systems given by Equation (14) in the (α,β,γ) −cut representation,

$$\frac{d}{dx}\left[\left(y_L(x,\alpha), y_R(x,\alpha)\right); \left(y'_L(x,\beta), y'_R(x,\beta)\right); \left(y''_L(x,\gamma), y''_R(x,\gamma)\right)\right]$$
$$= A\left[\left(z_L(x,\alpha), z_R(x,\alpha)\right); \left(z'_L(x,\beta), z'_R(x,\beta)\right); \left(z''_L(x,\gamma), z''_R(x,\gamma)\right)\right] \tag{16}$$

$$\frac{d}{dx}\left[\left(z_L(x,\alpha), z_R(x,\alpha)\right); \left(z'_L(x,\beta), z'_R(x,\beta)\right); \left(z''_L(x,\gamma), z''_R(x,\gamma)\right)\right]$$
$$= B\left[\left(y_L(x,\alpha), y_R(x,\alpha)\right); \left(y'_L(x,\beta), y'_R(x,\beta)\right); \left(y''_L(x,\gamma), y''_R(x,\gamma)\right)\right] \tag{17}$$

Case 1: When the coefficients$A$and $B$are positive numbers. We consider two distinct subcases under this instance in the manner described below:

Subcase 1.1: When $\tilde{y}_{Neu}$and $\tilde{z}_{Neu}$ are of type I differentiable.

Then, from Equations (16) and (17), we get,

$$\begin{cases} \frac{dy_L(x,\alpha)}{dx} = Az_L(x,\alpha) \\ \frac{dy_R(x,\alpha)}{dx} = Az_R(x,\alpha) \\ \frac{dy_L'(x,\beta)}{dx} = Az_L'(x,\beta) \\ \frac{dy_R'(x,\beta)}{dx} = Az_R'(x,\beta) \\ \frac{dy_L''(x,\gamma)}{dx} = Az_L''(x,\gamma) \\ \frac{dy_R''(x,\gamma)}{dx} = Az_R''(x,\gamma) \end{cases} \tag{18}$$

and

$$\begin{cases} \frac{dz_L(x,\alpha)}{dx} = By_L(x,\alpha) \\ \frac{dz_L(x,\alpha)}{dx} = By_R(x,\alpha) \\ \frac{dz_L'(x,\beta)}{dx} = By_L'(x,\beta) \\ \frac{dz_R'(x,\beta)}{dx} = By_R'(x,\beta) \\ \frac{dz_L''(x,\gamma)}{dx} = By_L''(x,\gamma) \\ \frac{dz_R''(x,\gamma)}{dx} = By_R''(x,\gamma) \end{cases} \tag{19}$$

Then,

$$\frac{d^2y_L(x,\alpha)}{dx^2} = ABy_L(x,\alpha) \tag{20}$$

which gives $y_L(x,\alpha) = c_1e^{\sqrt{AB}x} + c_2e^{-\sqrt{AB}x}$ and then, using the initial conditions we get

$$y_L(x,\alpha) = \frac{1}{2}\left\{y_{0L}(\alpha) + \sqrt{\frac{A}{B}}z_{0L}(\alpha)\right\}e^{\sqrt{AB}x} + \frac{1}{2}\left\{y_{0L}(\alpha) - \sqrt{\frac{A}{B}}z_{0L}(\alpha)\right\}e^{-\sqrt{AB}x} \tag{21}$$

and

$$z_L(x,\alpha) = \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0L}(\alpha) + \sqrt{\frac{A}{B}}z_{0L}(\alpha)\right\}e^{\sqrt{AB}x} - \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0L}(\alpha) - \sqrt{\frac{A}{B}}z_{0L}(\alpha)\right\}e^{-\sqrt{AB}x} \tag{22}$$

Similarly,

$$y_R(x,\alpha) = \frac{1}{2}\left\{y_{0R}(\alpha) + \sqrt{\frac{A}{B}}z_{0R}(\alpha)\right\}e^{\sqrt{AB}x} + \frac{1}{2}\left\{y_{0R}(\alpha) - \sqrt{\frac{A}{B}}z_{0R}(\alpha)\right\}e^{-\sqrt{AB}x} \tag{23}$$

$$z_R(x,\alpha) = \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0R}(\alpha) + \sqrt{\frac{A}{B}}z_{0R}(\alpha)\right\}e^{\sqrt{AB}x} - \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0R}(\alpha) - \sqrt{\frac{A}{B}}z_{0R}(\alpha)\right\}e^{-\sqrt{AB}x} \tag{24}$$

$$y_L'(x,\beta) = \frac{1}{2}\left\{y_{0L}'(\beta) + \sqrt{\frac{A}{B}}z_{0L}'(\beta)\right\}e^{\sqrt{AB}x} + \frac{1}{2}\left\{y_{0L}'(\beta) - \sqrt{\frac{A}{B}}z_{0L}'(\beta)\right\}e^{-\sqrt{AB}x} \tag{25}$$

$$z_L'(x,\beta) = \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0L}'(\beta) + \sqrt{\frac{A}{B}}z_{0L}'(\beta)\right\}e^{\sqrt{AB}x} - \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0L}'(\beta) - \sqrt{\frac{A}{B}}z_{0L}'(\beta)\right\}e^{-\sqrt{AB}x} \tag{26}$$

$$y_R'(x,\beta) = \frac{1}{2}\left\{y_{0R}'(\beta) + \sqrt{\frac{A}{B}}z_{0R}'(\beta)\right\}e^{\sqrt{AB}x} + \frac{1}{2}\left\{y_{0R}'(\beta) - \sqrt{\frac{A}{B}}z_{0R}'(\beta)\right\}e^{-\sqrt{AB}x} \tag{27}$$

$$z_R'(x,\beta) = \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0R}'(\beta) + \sqrt{\frac{A}{B}}z_{0R}'(\beta)\right\}e^{\sqrt{AB}x} - \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0R}'(\beta) - \sqrt{\frac{A}{B}}z_{0R}'(\beta)\right\}e^{-\sqrt{AB}x} \tag{28}$$

$$y_L''(x,\gamma) = \frac{1}{2}\left\{y_{0L}''(\gamma) + \sqrt{\frac{A}{B}}z_{0L}''(\gamma)\right\}e^{\sqrt{AB}x} + \frac{1}{2}\left\{y_{0L}''(\gamma) - \sqrt{\frac{A}{B}}z_{0L}''(\gamma)\right\}e^{-\sqrt{AB}x} \tag{29}$$

$$z_L''(x,\gamma) = \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0L}''(\gamma) + \sqrt{\frac{A}{B}}z_{0L}''(\gamma)\right\}e^{\sqrt{AB}x} - \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0L}''(\gamma) - \sqrt{\frac{A}{B}}z_{0L}''(\gamma)\right\}e^{-\sqrt{AB}x} \tag{30}$$

$$y_R''(x,\gamma) = \frac{1}{2}\left\{y_{0R}''(\gamma) + \sqrt{\frac{A}{B}}z_{0R}''(\gamma)\right\}e^{\sqrt{AB}x} + \frac{1}{2}\left\{y_{0R}''(\gamma) - \sqrt{\frac{A}{B}}z_{0R}''(\gamma)\right\}e^{-\sqrt{AB}x} \tag{31}$$

$$z_R''(x,\gamma) = \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0R}''(\gamma) + \sqrt{\frac{A}{B}}z_{0R}''(\gamma)\right\}e^{\sqrt{AB}x} - \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0R}''(\gamma) - \sqrt{\frac{A}{B}}z_{0R}''(\gamma)\right\}e^{-\sqrt{AB}x} \tag{32}$$

Subcase 1.2: When$\tilde{y}_{Neu}$and $\tilde{z}_{Neu}$ are of type II differentiable.
Then, from the Equations (16) and (17) we get,

$$\begin{cases} \frac{dy_R(x,\alpha)}{dx} = Az_L(x,\alpha) \\ \frac{dy_L(x,\alpha)}{dx} = Az_R(x,\alpha) \\ \frac{dy_R'(x,\beta)}{dx} = Az_L'(x,\beta) \\ \frac{dy_L'(x,\beta)}{dx} = Az_R'(x,\beta) \\ \frac{dy_R''(x,\gamma)}{dx} = Az_L''(x,\gamma) \\ \frac{dy_L''(x,\gamma)}{dx} = Az_R''(x,\gamma) \end{cases} \tag{33}$$

and

$$\begin{cases} \frac{dz_R(x,\alpha)}{dx} = By_L(x,\alpha) \\ \frac{dz_L(x,\alpha)}{dx} = By_R(x,\alpha) \\ \frac{dz_R'(x,\beta)}{dx} = By_L'(x,\beta) \\ \frac{dz_L'(x,\beta)}{dx} = By_R'(x,\beta) \\ \frac{dz_R''(x,\gamma)}{dx} = By_L''(x,\gamma) \\ \frac{dz_L''(x,\gamma)}{dx} = By_R''(x,\gamma) \end{cases} \tag{34}$$

Then,

$$\frac{d^2y_R(x,\alpha)}{dx^2} = ABy_R(x,\alpha) \tag{35}$$

which gives $y_R(x,\alpha) = c_3e^{\sqrt{AB}t} + c_4e^{-\sqrt{AB}t}$ and then, using the initial conditions we get

$$y_R(x,\alpha) = \frac{1}{2}\left\{y_{0R}(\alpha) + \sqrt{\frac{A}{B}}z_{0L}(\alpha)\right\}e^{\sqrt{AB}t} + \frac{1}{2}\left\{y_{0R}(\alpha) - \sqrt{\frac{A}{B}}z_{0L}(\alpha)\right\}e^{-\sqrt{AB}t} \tag{36}$$

and

$$z_L(x,\alpha) = \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0R}(\alpha) + \sqrt{\frac{A}{B}}z_{0L}(\alpha)\right\}e^{\sqrt{AB}t} - \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0R}(\alpha) - \sqrt{\frac{A}{B}}z_{0L}(\alpha)\right\}e^{-\sqrt{AB}t} \tag{37}$$

Similarly,

$$y_L(x,\alpha) = \frac{1}{2}\left\{y_{0L}(\alpha) + \sqrt{\frac{A}{B}}z_{0R}(\alpha)\right\}e^{\sqrt{AB}x} + \frac{1}{2}\left\{y_{0L}(\alpha) - \sqrt{\frac{A}{B}}z_{0R}(\alpha)\right\}e^{-\sqrt{AB}x} \tag{38}$$

$$z_R(x,\alpha) = \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0L}(\alpha) + \sqrt{\frac{A}{B}}z_{0R}(\alpha)\right\}e^{\sqrt{AB}x} - \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0L}(\alpha) - \sqrt{\frac{A}{B}}z_{0R}(\alpha)\right\}e^{-\sqrt{AB}x} \tag{39}$$

$$y_R'(x,\beta) = \frac{1}{2}\left\{y_{0R}'(\beta) + \sqrt{\frac{A}{B}}z_{0L}'(\beta)\right\}e^{\sqrt{AB}x} + \frac{1}{2}\left\{y_{0R}'(\beta) - \sqrt{\frac{A}{B}}z_{0L}'(\beta)\right\}e^{-\sqrt{AB}x} \tag{40}$$

$$z'_L(x,\beta) = \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y'_{0R}(\beta) + \sqrt{\frac{A}{B}}z'_{0L}(\beta)\right\}e^{\sqrt{AB}x} - \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y'_{0R}(\beta) - \sqrt{\frac{A}{B}}z'_{0L}(\beta)\right\}e^{-\sqrt{AB}x} \tag{41}$$

$$y'_L(x,\beta) = \frac{1}{2}\left\{y'_{0L}(\beta) + \sqrt{\frac{A}{B}}z'_{0R}(\beta)\right\}e^{\sqrt{AB}x} + \frac{1}{2}\left\{y'_{0L}(\beta) - \sqrt{\frac{A}{B}}z'_{0R}(\beta)\right\}e^{-\sqrt{AB}x} \tag{42}$$

$$z'_R(x,\beta) = \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y'_{0L}(\beta) + \sqrt{\frac{A}{B}}z'_{0R}(\beta)\right\}e^{\sqrt{AB}x} - \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y'_{0L}(\beta) - \sqrt{\frac{A}{B}}z'_{0R}(\beta)\right\}e^{-\sqrt{AB}x} \tag{43}$$

$$y''_R(x,\gamma) = \frac{1}{2}\left\{y''_{0R}(\gamma) + \sqrt{\frac{A}{B}}z''_{0L}(\gamma)\right\}e^{\sqrt{AB}x} + \frac{1}{2}\left\{y''_{0R}(\gamma) - \sqrt{\frac{A}{B}}z''_{0L}(\gamma)\right\}e^{-\sqrt{AB}x} \tag{44}$$

$$z''_L(x,\gamma) = \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y''_{0R}(\gamma) + \sqrt{\frac{A}{B}}z''_{0L}(\gamma)\right\}e^{\sqrt{AB}x} - \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y''_{0R}(\gamma) - \sqrt{\frac{A}{B}}z''_{0L}(\gamma)\right\}e^{-\sqrt{AB}x} \tag{45}$$

$$y''_L(x,\gamma) = \frac{1}{2}\left\{y''_{0L}(\gamma) + \sqrt{\frac{A}{B}}z''_{0R}(\gamma)\right\}e^{\sqrt{AB}x} + \frac{1}{2}\left\{y''_{0L}(\gamma) - \sqrt{\frac{A}{B}}z''_{0R}(\gamma)\right\}e^{-\sqrt{AB}x} \tag{46}$$

$$z''_R(x,\gamma) = \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y''_{0L}(\gamma) + \sqrt{\frac{A}{B}}z''_{0R}(\gamma)\right\}e^{\sqrt{AB}x} - \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y''_{0L}(\gamma) - \sqrt{\frac{A}{B}}z''_{0R}(\gamma)\right\}e^{-\sqrt{AB}x} \tag{47}$$

Case 2: When the coefficients A and B are negative numbers. We consider two distinct subcases under this instance in the manner described below:

Subcase 2.1: When $\tilde{y}_{Neu}$and $\tilde{z}_{Neu}$ are of type I differentiable.

Then, from the Equations (16) and (17) we get,

$$\begin{cases} \frac{dy_L(x,\alpha)}{dx} = Az_R(x,\alpha) \\ \frac{dy_R(x,\alpha)}{dx} = Az_L(x,\alpha) \\ \frac{dy'_L(x,\beta)}{dx} = Az'_R(x,\beta) \\ \frac{dy'_R(x,\beta)}{dx} = Az'_L(x,\beta) \\ \frac{dy''_L(x,\gamma)}{dx} = Az''_R(x,\gamma) \\ \frac{dy''_R(x,\gamma)}{dx} = Az''_L(x,\gamma) \end{cases} \tag{48}$$

and

$$\begin{cases} \frac{dz_L(x,\alpha)}{dx} = By_R(x,\alpha) \\ \frac{dz_L(x,\alpha)}{dx} = By_L(x,\alpha) \\ \frac{dz'_L(x,\beta)}{dx} = By'_R(x,\beta) \\ \frac{dz'_R(x,\beta)}{dx} = By'_L(x,\beta) \\ \frac{dz''_L(x,\gamma)}{dx} = By''_R(x,\gamma) \\ \frac{dz''_R(x,\gamma)}{dx} = By''_L(x,\gamma) \end{cases} \tag{49}$$

Then,

$$\frac{d^2y_L(x,\alpha)}{dx^2} = ABy_L(x,\alpha) \tag{50}$$

which gives $y_L(x,\alpha) = c_5e^{\sqrt{AB}t} + c_6e^{-\sqrt{AB}t}$ and then, using the initial conditions, we get

$$y_L(x,\alpha) = \frac{1}{2}\left\{y_{0L}(\alpha) + \sqrt{\frac{A}{B}}z_{0R}(\alpha)\right\}e^{\sqrt{AB}x} + \frac{1}{2}\left\{y_{0L}(\alpha) - \sqrt{\frac{A}{B}}z_{0R}(\alpha)\right\}e^{-\sqrt{AB}x} \tag{51}$$

$$z_R(x,\alpha)=\frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0L}(\alpha)+\sqrt{\frac{A}{B}}z_{0R}(\alpha)\right\}e^{\sqrt{AB}x}-\frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0L}(\alpha)-\sqrt{\frac{A}{B}}z_{0R}(\alpha)\right\}e^{-\sqrt{AB}x} \quad (52)$$

Similarly,

$$y_R(x,\alpha)=\frac{1}{2}\left\{y_{0R}(\alpha)+\sqrt{\frac{A}{B}}z_{0L}(\alpha)\right\}e^{\sqrt{AB}t}+\frac{1}{2}\left\{y_{0R}(\alpha)-\sqrt{\frac{A}{B}}z_{0L}(\alpha)\right\}e^{-\sqrt{AB}t} \quad (53)$$

$$z_L(x,\alpha)=\frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0R}(\alpha)+\sqrt{\frac{A}{B}}z_{0L}(\alpha)\right\}e^{\sqrt{AB}t}-\frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0R}(\alpha)-\sqrt{\frac{A}{B}}z_{0L}(\alpha)\right\}e^{-\sqrt{AB}t} \quad (54)$$

$$y'_L(x,\beta)=\frac{1}{2}\left\{y'_{0L}(\beta)+\sqrt{\frac{A}{B}}z'_{0R}(\beta)\right\}e^{\sqrt{AB}x}+\frac{1}{2}\left\{y'_{0L}(\beta)-\sqrt{\frac{A}{B}}z'_{0R}(\beta)\right\}e^{-\sqrt{AB}x} \quad (55)$$

$$z'_R(x,\beta)=\frac{1}{2}\sqrt{\frac{B}{A}}\left\{y'_{0L}(\beta)+\sqrt{\frac{A}{B}}z'_{0R}(\beta)\right\}e^{\sqrt{AB}x}-\frac{1}{2}\sqrt{\frac{B}{A}}\left\{y'_{0L}(\beta)-\sqrt{\frac{A}{B}}z'_{0R}(\beta)\right\}e^{-\sqrt{AB}x} \quad (56)$$

$$y'_R(x,\beta)=\frac{1}{2}\left\{y'_{0R}(\beta)+\sqrt{\frac{A}{B}}z'_{0L}(\beta)\right\}e^{\sqrt{AB}x}+\frac{1}{2}\left\{y'_{0R}(\beta)-\sqrt{\frac{A}{B}}z'_{0L}(\beta)\right\}e^{-\sqrt{AB}x} \quad (57)$$

$$z'_L(x,\beta)=\frac{1}{2}\sqrt{\frac{B}{A}}\left\{y'_{0R}(\beta)+\sqrt{\frac{A}{B}}z'_{0L}(\beta)\right\}e^{\sqrt{AB}x}-\frac{1}{2}\sqrt{\frac{B}{A}}\left\{y'_{0R}(\beta)-\sqrt{\frac{A}{B}}z'_{0L}(\beta)\right\}e^{-\sqrt{AB}x} \quad (58)$$

$$y''_L(x,\gamma)=\frac{1}{2}\left\{y''_{0L}(\gamma)+\sqrt{\frac{A}{B}}z''_{0R}(\gamma)\right\}e^{\sqrt{AB}x}+\frac{1}{2}\left\{y''_{0L}(\gamma)-\sqrt{\frac{A}{B}}z''_{0R}(\gamma)\right\}e^{-\sqrt{AB}x} \quad (59)$$

$$z''_R(x,\gamma)=\frac{1}{2}\sqrt{\frac{B}{A}}\left\{y''_{0L}(\gamma)+\sqrt{\frac{A}{B}}z''_{0R}(\gamma)\right\}e^{\sqrt{AB}x}-\frac{1}{2}\sqrt{\frac{B}{A}}\left\{y''_{0L}(\gamma)-\sqrt{\frac{A}{B}}z''_{0R}(\gamma)\right\}e^{-\sqrt{AB}x} \quad (60)$$

$$y''_R(x,\gamma)=\frac{1}{2}\left\{y''_{0R}(\gamma)+\sqrt{\frac{A}{B}}z''_{0L}(\gamma)\right\}e^{\sqrt{AB}x}+\frac{1}{2}\left\{y''_{0R}(\gamma)-\sqrt{\frac{A}{B}}z''_{0L}(\gamma)\right\}e^{-\sqrt{AB}x} \quad (61)$$

$$z''_L(x,\gamma)=\frac{1}{2}\sqrt{\frac{B}{A}}\left\{y''_{0R}(\gamma)+\sqrt{\frac{A}{B}}z''_{0L}(\gamma)\right\}e^{\sqrt{AB}x}-\frac{1}{2}\sqrt{\frac{B}{A}}\left\{y''_{0R}(\gamma)-\sqrt{\frac{A}{B}}z''_{0L}(\gamma)\right\}e^{-\sqrt{AB}x} \quad (62)$$

Subcase 2.2: When $\tilde{y}_{Neu}$and $\tilde{z}_{Neu}$ are of type II differentiable.
Then, from the Equations (16) and (17) we get,

$$\begin{cases}\frac{dy_R(x,\alpha)}{dx}=Az_R(x,\alpha)\\ \frac{dy_L(x,\alpha)}{dx}=Az_L(x,\alpha)\\ \frac{dy'_R(x,\beta)}{dx}=Az'_R(x,\beta)\\ \frac{dy'_L(x,\beta)}{dx}=Az'_L(x,\beta)\\ \frac{dy''_R(x,\gamma)}{dx}=Az''_R(x,\gamma)\\ \frac{dy''_L(x,\gamma)}{dx}=Az''_L(x,\gamma)\end{cases} \quad (63)$$

and

$$\begin{cases} \frac{dz_R(x,\alpha)}{dx} = By_R(x,\alpha) \\ \frac{dz_L(x,\alpha)}{dx} = By_L(x,\alpha) \\ \frac{dz'_R(x,\beta)}{dx} = By'_R(x,\beta) \\ \frac{dz'_L(x,\beta)}{dx} = By'_L(x,\beta) \\ \frac{dz''_R(x,\gamma)}{dx} = By''_R(x,\gamma) \\ \frac{dz''_L(x,\gamma)}{dx} = By''_L(x,\gamma) \end{cases} \tag{64}$$

Then,

$$\frac{d^2y_R(x,\alpha)}{dx^2} = ABy_R(x,\alpha) \tag{65}$$

which gives $y_R(x,\alpha) = c_7e^{\sqrt{AB}x} + c_8e^{-\sqrt{AB}x}$ and then, using the initial conditions we get

$$y_R(x,\alpha) = \frac{1}{2}\left\{y_{0R}(\alpha) + \sqrt{\frac{A}{B}}z_{0R}(\alpha)\right\}e^{\sqrt{AB}x} + \frac{1}{2}\left\{y_{0R}(\alpha) - \sqrt{\frac{A}{B}}z_{0R}(\alpha)\right\}e^{-\sqrt{AB}x} \tag{66}$$

and

$$z_R(x,\alpha) = \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0R}(\alpha) + \sqrt{\frac{A}{B}}z_{0R}(\alpha)\right\}e^{\sqrt{AB}x} - \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0R}(\alpha) - \sqrt{\frac{A}{B}}z_{0R}(\alpha)\right\}e^{-\sqrt{AB}x} \tag{67}$$

Similarly,

$$y_L(x,\alpha) = \frac{1}{2}\left\{y_{0L}(\alpha) + \sqrt{\frac{A}{B}}z_{0L}(\alpha)\right\}e^{\sqrt{AB}x} + \frac{1}{2}\left\{y_{0L}(\alpha) - \sqrt{\frac{A}{B}}z_{0L}(\alpha)\right\}e^{-\sqrt{AB}x} \tag{68}$$

$$z_L(x,\alpha) = \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0L}(\alpha) + \sqrt{\frac{A}{B}}z_{0L}(\alpha)\right\}e^{\sqrt{AB}x} - \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y_{0L}(\alpha) - \sqrt{\frac{A}{B}}z_{0L}(\alpha)\right\}e^{-\sqrt{AB}x} \tag{69}$$

$$y'_R(x,\beta) = \frac{1}{2}\left\{y'_{0R}(\beta) + \sqrt{\frac{A}{B}}z'_{0R}(\beta)\right\}e^{\sqrt{AB}x} + \frac{1}{2}\left\{y'_{0R}(\beta) - \sqrt{\frac{A}{B}}z'_{0R}(\beta)\right\}e^{-\sqrt{AB}x} \tag{70}$$

$$z'_R(x,\beta) = \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y'_{0R}(\beta) + \sqrt{\frac{A}{B}}z'_{0R}(\beta)\right\}e^{\sqrt{AB}x} - \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y'_{0R}(\beta) - \sqrt{\frac{A}{B}}z'_{0R}(\beta)\right\}e^{-\sqrt{AB}x} \tag{71}$$

$$y'_L(x,\beta) = \frac{1}{2}\left\{y'_{0L}(\beta) + \sqrt{\frac{A}{B}}z'_{0L}(\beta)\right\}e^{\sqrt{AB}x} + \frac{1}{2}\left\{y'_{0L}(\beta) - \sqrt{\frac{A}{B}}z'_{0L}(\beta)\right\}e^{-\sqrt{AB}x} \tag{72}$$

$$z'_L(x,\beta) = \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y'_{0L}(\beta) + \sqrt{\frac{A}{B}}z'_{0L}(\beta)\right\}e^{\sqrt{AB}x} - \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y'_{0L}(\beta) - \sqrt{\frac{A}{B}}z'_{0L}(\beta)\right\}e^{-\sqrt{AB}x} \tag{73}$$

$$y''_R(x,\gamma) = \frac{1}{2}\left\{y''_{0R}(\gamma) + \sqrt{\frac{A}{B}}z''_{0R}(\gamma)\right\}e^{\sqrt{AB}x} + \frac{1}{2}\left\{y''_{0R}(\gamma) - \sqrt{\frac{A}{B}}z''_{0R}(\gamma)\right\}e^{-\sqrt{AB}x} \tag{74}$$

$$z''_R(x,\gamma) = \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y''_{0R}(\gamma) + \sqrt{\frac{A}{B}}z''_{0R}(\gamma)\right\}e^{\sqrt{AB}x} - \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y''_{0R}(\gamma) - \sqrt{\frac{A}{B}}z''_{0R}(\gamma)\right\}e^{-\sqrt{AB}x} \tag{75}$$

$$y''_L(x,\gamma) = \frac{1}{2}\left\{y''_{0L}(\gamma) + \sqrt{\frac{A}{B}}z''_{0L}(\gamma)\right\}e^{\sqrt{AB}x} + \frac{1}{2}\left\{y''_{0L}(\gamma) - \sqrt{\frac{A}{B}}z''_{0L}(\gamma)\right\}e^{-\sqrt{AB}x} \tag{76}$$

$$z''_L(x,\gamma) = \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y''_{0L}(\gamma) + \sqrt{\frac{A}{B}}z''_{0L}(\gamma)\right\}e^{\sqrt{AB}x} - \frac{1}{2}\sqrt{\frac{B}{A}}\left\{y''_{0L}(\gamma) - \sqrt{\frac{A}{B}}z''_{0L}(\gamma)\right\}e^{-\sqrt{AB}x} \tag{77}$$

4. Application

Consider the arm race model [31] in a Neutrosophic arena. Suppose two conflicting nations A and B are there with their capacities of $\tilde{y}_{Neu}(t)$ and $\tilde{Z}_{Neu}(t)$ number armaments respectively at time t.

One nation increases their armaments in a competitive proportion with another nation to defend itself from the possible aggression of the opponent. Now, the initial information about the strengths of two opponent countries may be imprecise to each other and they can try to predict it using the philosophy of neutrosophy. When the available information is considered about a value in bell-shaped generalized sense, the Cauchy neutrosophic number can fulfill the purposes. Consequently, the following system of Neutrosophic linear homogeneous differential equations can be applied to stand for the system.

$$\begin{cases} \frac{d\tilde{y}_{Neu}(t)}{dt} = M\ \tilde{z}_{Neu}(t) \\ \frac{d\tilde{z}_{Neu}(t)}{dx} = N\ \tilde{y}_{Neu}(t) \end{cases} \tag{78}$$

with the initial conditions

$$\begin{cases} \tilde{y}_{Neu}(0) = \tilde{y}_{0\ Neu} \\ \tilde{z}_{Neu}(0) = \tilde{z}_{0\ Neu} \end{cases} \tag{79}$$

In our numerical simulation, we assume that the efficiency to increase their armaments are equal for both nations. So, let $M = N = 0.5$.

Also, let $\tilde{y}_{0\ Neu}(0) =< (1; 40), (1.5; 40), (2; 40)$ and $\tilde{z}_{0\ Neu}(0) =< (1; 0); (1.5,0), (2,0) >$ be two Cauchy Neutrosohphic numbers. Figure 1 and Figure 2 visualize the Cauchy Neutrosophic numbers $\tilde{y}_{0\ Neu}(0)$ and $\tilde{z}_{0\ Neu}(0)$, respectively.

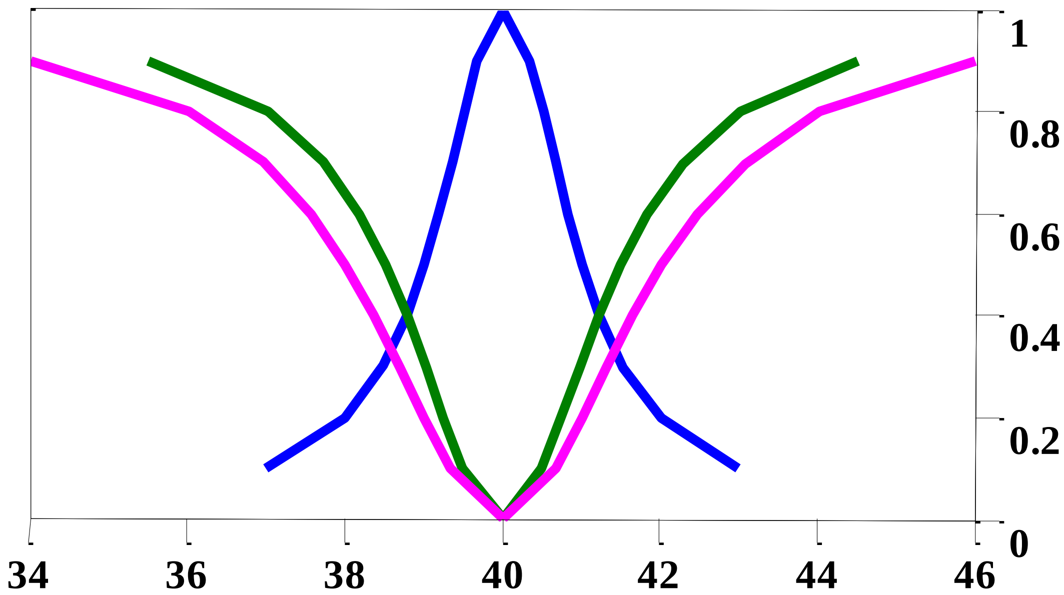

Fig. 1. Initial armament $\tilde{y}_{0\ Neu}(0)$ of the nation A

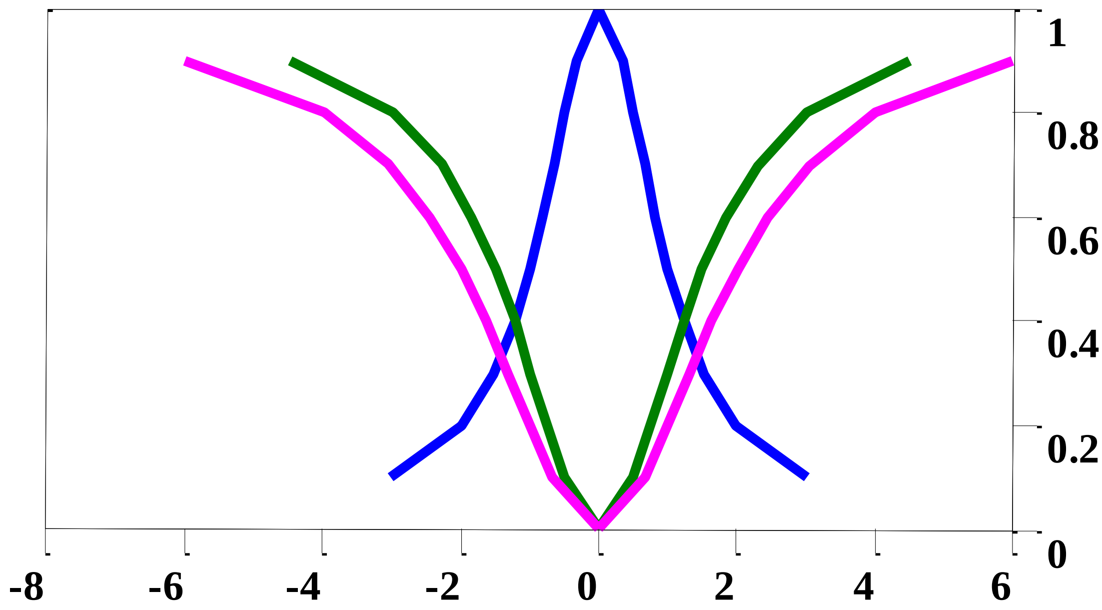

Fig. 2. Initial armament $\tilde{z}_{0\ Neu}(0)$ of the nation B

The parametric forms of the Cauchy neutrosophic numbers are given by

$$\tilde{y}_{Neu}(0) = \left[\left(y_{0L}(\alpha), y_{0R}(\alpha)\right); \left(y'_{0L}(\beta), y'_{0R}(\beta)\right); \left(y''_{0L}(\gamma), y''_{0R}(\gamma)\right)\right] \tag{80}$$

and

$$\tilde{z}_{Neu}(0) = \left[\left(z_{0L}(\alpha), z_{0R}(\alpha)\right); \left(z'_{0L}(\beta), z'_{0R}(\beta)\right); \left(z''_{0L}(\gamma), z''_{0R}(\gamma)\right)\right] \tag{81}$$

where

$$\begin{cases} y_{0L}(\alpha) = 40 - \sqrt{\frac{1-\alpha}{\alpha}} \\ y_{0R}(\alpha) = 40 + \sqrt{\frac{1-\alpha}{\alpha}} \\ y'_{0L}(\beta) = 40 - 1.5\sqrt{\frac{\beta}{1-\beta}} \\ y'_{0R}(\beta) = 40 + 1.5\sqrt{\frac{\beta}{1-\beta}} \\ y''_{0L}(\gamma) = 40 - 2\sqrt{\frac{\gamma}{1-\gamma}} \\ y''_{0R}(\gamma) = 40 + 2\sqrt{\frac{\gamma}{1-\gamma}} \end{cases} \tag{82}$$

and

$$\begin{cases} z_{0L}(\alpha) = -\sqrt{\frac{1-\alpha}{\alpha}} \\ z_{0R}(\alpha) = \sqrt{\frac{1-\alpha}{\alpha}} \\ z_{0L}^{'}(\beta) = -1.5\sqrt{\frac{\beta}{1-\beta}} \\ z_{0R}^{'}(\beta) = 1.5\sqrt{\frac{\beta}{1-\beta}} \\ z_{0L}^{''}(\gamma) = -2\sqrt{\frac{\gamma}{1-\gamma}} \\ z_{0R}^{''}(\gamma) = 2\sqrt{\frac{\gamma}{1-\gamma}} \end{cases} \tag{83}$$

Then, taking the differentiability of $\tilde{y}_{Neu}$and $\tilde{z}_{Neu}$ are of type I, the solution of the system given by Equation (78) is obtained as

$$\begin{cases} y_L(t,\alpha) = \left(20 - \sqrt{\frac{1-\alpha}{\alpha}}\right) e^{0.5t} + 20e^{-0.5t} \\ z_L(t,\alpha) = \left(20 - \sqrt{\frac{1-\alpha}{\alpha}}\right) e^{0.5t} - 20e^{-0.5t} \\ y_R(t,\alpha) = \left(20 + \sqrt{\frac{1-\alpha}{\alpha}}\right) e^{0.5t} + 20e^{-0.5t} \\ z_R(t,\alpha) = \left(20 + \sqrt{\frac{1-\alpha}{\alpha}}\right) e^{0.5t} - 20e^{-0.5t} \\ y_L'(t,\beta) = \left(20 - 1.5\sqrt{\frac{\beta}{1-\beta}}\right) e^{0.5t} + 20e^{-0.5t} \\ z_L'(t,\beta) = \left(20 - 1.5\sqrt{\frac{\beta}{1-\beta}}\right) e^{0.5t} - 20e^{-0.5t} \\ y_R'(t,\beta) = \left(20 + 1.5\sqrt{\frac{\beta}{1-\beta}}\right) e^{0.5t} + 20e^{-0.5t} \\ z_R'(t,\beta) = \left(20 + 1.5\sqrt{\frac{\beta}{1-\beta}}\right) e^{0.5t} - 20e^{-0.5t} \\ y_L''(t,\gamma) = \left(20 - 2\sqrt{\frac{\gamma}{1-\gamma}}\right) e^{0.5t} + 20e^{-0.5t} \\ z_L''(t,\gamma) = \left(20 - 2\sqrt{\frac{\gamma}{1-\gamma}}\right) e^{0.5t} - 20e^{-0.5t} \\ y_R''(t,\gamma) = \left(20 + 2\sqrt{\frac{\gamma}{1-\gamma}}\right) e^{0.5t} + 20e^{-0.5t} \\ y_R''(t,\gamma) = \left(20 + 2\sqrt{\frac{\gamma}{1-\gamma}}\right) e^{0.5t} - 20e^{-0.5t} \end{cases} \tag{84}$$

Then, after 5 years of the arm race, components of the parametric representation of the uncertain armaments of nation A are represented in Table 1 for different levels of aspirations.

Table 1
Armaments of the nation A after 5 years

α,β,γ	$y_L(t,\alpha)$	$y_R(t,\alpha)$	$y_L'(t,\beta)$	$y_R'(t,\beta)$	$y_L''(t,\gamma)$	$y_R''(t,\gamma)$
0	$-\infty$	$+\infty$	245.2916	245.2916	245.29158	245.2916
0.1	208.744098	281.83906	239.2003	251.3828	237.16992	253.4132
0.2	220.926592	269.65657	236.154	254.4285	233.10909	257.4741
0.3	226.682513	263.900647	233.3286	257.2546	229.34095	261.2422
0.4	230.3711329	260.212027	230.3711	260.212	225.39765	265.1855
0.5	233.109086	257.474074	227.0178	263.5653	220.92659	269.6566
0.6	235.3446153	255.238545	222.9109	267.6723	215.45069	275.1325
0.7	237.3162656	253.266894	217.378	273.2052	208.07345	282.5097
0.8	239.200333	251.382827	208.7441	281.8391	196.5616	294.0216
0.9	241.2307486	249.352411	190.47036	300.1128	172.19662	318.3865
1	245.29158	245.29158	$-\infty$	$+\infty$	$-\infty$	$+\infty$

The tabular data from Table 1 can be put into the graphical representation in Figure 3, which can interpret the situation more precisely.

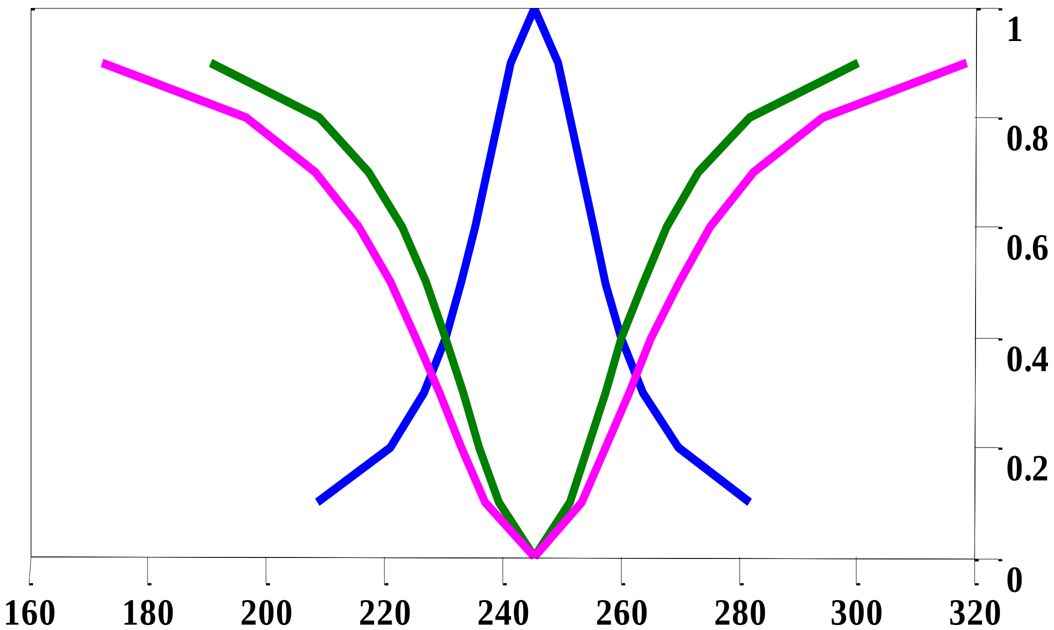

Fig. 3. Armaments $\tilde{y}_{Neu}(t)$ of the nation A after 5 years

Then, after 5 years of the arm race, components of the parametric representation of the uncertain armaments of nation B are represented in Table 2 for different levels of aspirations.

Table 2
Armaments of the nation B after 5 years

α, β, γ	$z_L(t,\alpha)$	$z_R(t,\alpha)$	$z'_L(t,\beta)$	$z'_R(t,\beta)$	$z''_L(t,\gamma)$	$z''_R(t,\gamma)$
0	$-\infty$	$+\infty$	242.0082	242.0082	242.0082	242.0082
0.1	205.460698	278.55566	235.9169	248.0994	233.8865	250.1298
0.2	217.643192	266.37317	232.8713	251.1451	229.8257	254.1907
0.3	223.3991131	260.61725	230.0452	253.9712	226.0576	257.9588
0.4	227.087733	256.92863	227.0877	256.9286	222.1143	261.9021
0.5	229.825686	254.19067	223.7344	260.2819	217.6432	266.3732
0.6	232.0612153	251.95514	219.6275	264.3889	212.1673	271.8491
0.7	234.0328656	249.98349	214.0946	269.9218	204.79	279.2263
0.8	235.916933	248.09943	205.4607	278.5557	193.2782	290.7382
0.9	237.9473487	246.06901	187.187	296.8294	168.91322	315.1031
1	242.00818	242.00818	$-\infty$	$+\infty$	$-\infty$	$+\infty$

The tabular data from Table 2 can be put into the graphical representation in Figure 4 which can interpretate the situation more precisely.

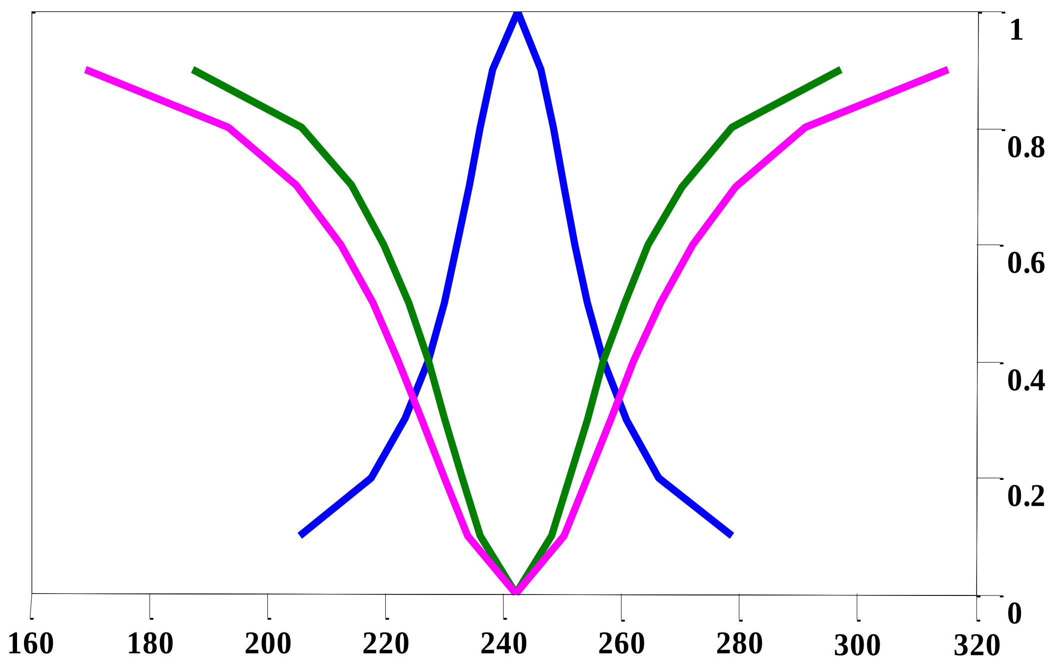

Fig. 4. Armaments $\tilde{z}_{Neu}(t)$ of the nation B after 5 years

Remark 1: The bell-shaped nature of the initial neutrosophic information is preserved in the solution also.

5. Conclusion

In this paper, a system of neutrosophic linear homogenous differential equations is analyzed in a neutrosophic environment. Here, the initial states are taken as the neutrosophic numbers and the manifestation is done based on the generalized differentiability of the neutrosophic valued functions. From our discussion in this chapter, it is perceived that the theory of system of differential equations in crisp and fuzzy environment can be easily extended into the domain of neutrosophic uncertainty which is more specific and structured sense of uncertainty. Also, the arms race phenomenon between two conflicting nations is aptly depicted in this chapter with Cauchy's neutrosophic numbers as the information about the initial armaments. In future, more theories on the uncertain system of differential equations can be developed in this direction. Also, the mathematical modelling of real physical phenomena as an application of the proposed theory may be a matter of future challenges in this context.

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Conflicts of Interest
The authors declare no conflicts of interest.

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