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Interval-value Pythagorean Fuzzy Prioritized Aggregation Operators for Selecting an Eco-Friendly Transportation Mode Selection

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ABSTRACT

When selecting an environmentally friendly mode of transportation, many factors must be taken into account, such as capacity, delivery time, cost-effectiveness, and environmental impact. This is a difficult decision-making problem since it calls for the simultaneous evaluation and prioritization of several criteria. A thorough and methodical approach is required to make an informed decision that satisfies particular transportation needs while adhering to sustainability goals. The process entails balancing various trade-offs to determine the most environmentally friendly mode of transportation. When solving multi-attribute group decision-making (MAGDM) problems, prioritization is essential. Prioritization in fuzzy systems has been implemented through a variety of techniques and approaches. This paper tackles the MAGDM problem within the Pythagorean fuzzy (PyF) framework, taking into account the different requirements for experts and characteristics. We present new Aczel Alsina aggregation operators (AOs), whose efficient handling of uncertainties makes a major contribution to fuzzy mathematics. We suggest several PyF AOs, such as the PyF-prioritized Aczel Alsina geometric (PyFPAAG) and PyF-prioritized Aczel Alsina averaging (PyFPAAA), which are based on the Aczel Alsina t-norm and t-conorm. We show these AOs satisfy the aggregation criteria by examining their monotonicity, boundedness, and idempotency properties. The prioritization weights are derived from expert knowledge, enabling the suggested operators to capture the prioritization phenomenon among the aggregated arguments. Using a MAGDM technique, the proposed AOs are used to evaluate eco-friendly transportation modes. Their significance is confirmed by contrasting them with other well-known AOs.

1. Introduction

When it comes to transportation, eco-friendly techniques are those that, when compared to conventional modes, drastically cut emissions, energy use, and pollution. Through the use of more resource-efficient technologies, renewable energy sources, and cleaner technologies, these approaches support sustainability. Typical instances comprise strolling, riding a bike, utilizing renewa-

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ble energy sources to run buses and trains, and drive electric cars (EVs). It is impossible to overestimate the importance of environmentally friendly transportation in the fight against climate change and environmental degradation. Conventional automobiles that run on fossil fuels have a significant impact on greenhouse gas emissions and air pollution. Societies can significantly lower their carbon footprints and lessen the negative effects of pollution by switching to environmentally friendly alternatives. It is critical to reduce harmful emissions, such as carbon dioxide (CO_2), nitrogen oxides (NOx), and particulate matter. For example, the absence of exhaust gases from electric cars contributes to better air quality and lower risk of diseases linked to pollution. Likewise, public transportation networks that run on renewable energy or cleaner fuels can reduce emissions from individual cars and traffic jams. Using eco-friendly transportation also encourages efficiency and energy savings. A more sustainable energy ecosystem is facilitated by the use of renewable energy sources, such as solar or wind power, to charge electric vehicles. Additionally, by addressing problems like obesity and sedentary behavior, modes like walking and cycling not only reduce emissions but also encourage healthier lifestyles. By lowering the need for large infrastructure, like parking lots and roadways, these transportation options support sustainable urban growth. Bike lanes and public transportation investments encourage compact, walkable neighborhoods, which decrease land use, preserve green spaces, and improve the general livability of urban areas. Ku et al. [1] assessed the impact of environmentally friendly transportation. Ajay et al. [2] created an IoT-based management system for environmentally friendly urban transportation. Kuzey et al. [3] talked about environmentally friendly transportation initiatives and the CSR approach. Alhamrani et al. [4] created the best environmentally friendly transportation system. Several benefits of environmentally friendly transportation were emphasized by Ku et al. [5]. Guidelines for environmentally friendly transportation infrastructure were given by Bencekri et al. [6]. Several environmentally friendly transportation options were investigated by Gao et al. [7].

Zadeh [8] developed the fuzzy set theory (FSs) in 1965 to deal with uncertain phenomena. While FSs use a membership grade (MG) to represent uncertainty, they ignore the non-membership grade (NMG), which is also important for human evaluation. In order to get around this restriction, Atanassov [9] created the idea of intuitionistic fuzzy sets (IFSs), which combine MG and NMG to provide a more thorough representation of ambiguous phenomena. Because the total of the MG and NMG cannot be greater than 1, pairs like (0.9, 0.7) are invalid as intuitionistic fuzzy values (IFVs) in IFSs. Furthermore, IFSs compute the hesitancy grade (HG), which is derived by deducting the sum of MG and NMG from 1. This allows them to quantify the hesitancy of human judgment. [10] Pythagorean fuzzy subsets (PyFSs) are a new evaluation format that Yager recently proposed to capture more meaningful information in imprecise and uncertain situations [11,12]. PyFSs are defined by the degrees of membership and non-membership that meet the requirement that the total squared of these degrees do not exceed 1 [12]. With the help of this framework, uncertainty can be represented more adaptable and decision-making in fuzzy environments can be done with greater freedom [13-14].

In fuzzy mathematics, Aczel-Alsina aggregation operators (AOs) have undergone substantial development [15-17]. When solving classification problems with different t-norms (TNs) and t-conorms (TCNs), Farahbod and Eftekhari [18] discovered that the Aczel-Alsina TN was particularly helpful. This paper presents prioritized AOs for intuitionistic fuzzy sets (IFS) based on Aczel-Alsina TN and TCN, given the importance of these AOs and the necessity of prioritization in aggregation processes. By using intuitionistic fuzzy values (IFVs), these AOs will make data aggregation easier while preserving a hierarchy of expert or attribute prioritization. Certain experts and attributes may be given priority in multi-attribute group decision-making (MAGDM) problems. Yager [19] first put forth the concept of prioritized aggregation operators (AOs) in 2008, emphasizing the significance of these entities in certain circumstances. Yan et al. [20] extended the prioritization research, while Yu and Xu [21] introduced prioritized AOs for intuitionistic fuzzy (IF) information. Prioritization for linguistic IF information aggregation was proposed by Arora and Garg [22], while Ali et al. [23] investigated it in the context of complex IF soft sets for MAGDM problems. Gao [24] used Einstein t-norms and t-conorms to create prioritized AOs for Pythagorean fuzzy information, while Chen [25] covered interval-valued prioritized AO comparison points for MAGDM issues. Prioritized AOs for bipolar fuzzy information were created by Jana et al. [26] using Dombi t-norms and t-conorms. Prioritized AOs for generalized orthopair fuzzy sets were studied by Riaz et al. [27].

The key features of the manuscript are as follows:

i. The introduction of IVPyFPAAG and IVPyFPAAA operators, which are prioritized AOs based on Aczel-Alsina TN and TCN in IVIF data.
ii. An analysis of the IVPyFPAAG and IVPyFPAAA operators that are being proposed.
iii. Examination of the suggested prioritized AOs' characteristics.
iv. MAGDM problem solved with the help of the IVPyFPAAG and IVPyFPAAA operators.
v. Comparison of the results with other ongoing AOs and the IVPyFVs' suggested prioritized AOs.

The work is organized as follows: Section 2 covers some foundational concepts in detail. The proposed interval-valued Pythagorean fuzzy prioritized Aczel-Alsina averaging (IVPyFPAAA) and interval-valued Pythagorean fuzzy prioritized Aczel-Alsina geometric (IVPyFPAAG) operators are introduced and their properties are examined in Section 3. In Section 4, an algorithm for MAGDM based on the IVPyFPAAG and IVPyFPAAA operators is presented. Section 5 provides a comprehensive example of the decision-making process. A comparison between the proposed techniques and established methods is given in Section 6. The article concludes with a summary and final thoughts in Section 7.

2. Methodology

Atanassov [9] introduced the concept of Interval-Valued Pythagorean Fuzzy Sets (IVPyFS) as an extension of fuzzy sets (FS). While fuzzy sets provide a membership grade (MG) indicating the degree of an element's membership in a set, Intuitionistic Fuzzy Sets (IFS) extend this by providing both a membership grade (MG) and a non-membership grade (NMG). For fuzzy sets, the MG is a real number between 0 and 1, and the same applies to the NMG. Additionally, the sum of the MG and NMG must be less than or equal to 1.

Definition 1 [9]. Let F be considered a universe of discourse, an IVPyFS in F is an expression ϱ given by:

$$\sigma = \left\{\left(\xi, [\psi_\sigma^l(\xi), \psi_\sigma^U(\xi)], [\mathcal{M}_\sigma^l(\xi), \mathcal{M}_\sigma^U(\xi)]\right) \mid \xi \in F\right\}, \tag{1}$$

where $[\psi_\sigma^l(\xi), \psi_\sigma^U(\xi)]: F \to [0,1]$ and $[\mathcal{M}_\sigma^l(\xi), \mathcal{M}_\sigma^U(\xi)]: F \to [0,1]$ including the condition $0 \leq \psi_\sigma^{2U}(\xi) + \mathcal{M}_\sigma^{2U}(\xi) \leq 1$ for each ξ in F. The intervals $\psi_\sigma^{2l}(\xi), \psi_\sigma^{2U}(\xi)$ and $\mathcal{M}_\sigma^{2l}(\xi), \mathcal{M}_\sigma^{2U}(\xi)$ serve as MD and NMD of the element ξ in the set F. For every IVPyFS σ in F, we denote $\pi_\sigma(\xi) = \left[\sqrt{\left(1 - \left(\psi_\sigma^{2U}(\xi) + \mathcal{M}_\sigma^{2U}(\xi)\right)\right)}, \left(\sqrt{1 - \left(\psi_\sigma^{2l}(\xi) + \mathcal{M}_\sigma^{2l}(\xi)\right)}\right)\right], \forall\, \xi \in F$. Then, $\pi_\sigma(\xi)$ is known as the hesitancy degree (HD) of ξ to σ. Further, $([\psi_\sigma^l(\xi), \psi_\sigma^U(\xi)], [\mathcal{M}_\sigma^l(\xi), \mathcal{M}_\sigma^U(\xi)])$ is known by Interval-value Pythagorean fuzzy value (IVPyFV).

Definition 2 [28]. Let $\sigma_1 = \left(\left[\psi^l_{\sigma_\tau}, \psi^U_{\sigma_\tau}\right], \left[\mathcal{M}^l_{\sigma_\tau}, \mathcal{M}^U_{\sigma_\tau}\right]\right)$ be an IVPyFV. Then, the score value of σ_1 is:

$$Sco(\sigma_1) = \frac{\psi^l_{\sigma_1} + \psi^U_{\sigma_1} - \mathcal{M}^l_{\sigma_1} - \mathcal{M}^U_{\sigma_1}}{2}, \tag{2}$$

and the degrees of accuracy of σ_1 is:

$$Acc(\sigma_1) = \frac{\psi^l_{\sigma_1} + \psi^U_{\sigma_1} + \mathcal{M}^l_{\sigma_1} + \mathcal{M}^U_{\sigma_1}}{2}. \tag{3}$$

Definition 3 [29]. The Aczel-Alsina t-norms $(T^{\circledcirc}_A)_{\circledcirc\epsilon[0,\infty]}$ is ascertained by:

$$(T^{\circledcirc}_A)_{(\ell,v)} = \begin{cases} T_D(\ell, v) & if\ \circledcirc = 0 \\ min(\ell, v) & if\ \circledcirc = \infty \\ exp^{-((-LN\ell)^{\circledcirc} + (-LNv)^{\circledcirc})^{\frac{1}{\circledcirc}}} & otherwise \end{cases}, \tag{4}$$

and the Aczel-Alsina t-conorms $(S^{\circledcirc}_A)_{\circledcirc\epsilon[0,\infty]}$ is ascertained by:

$$(S^{\circledcirc}_A)_{(\ell,v)} = \begin{cases} S_D(\ell, v) & if\ \circledcirc = 0 \\ max(\ell, v) & if\ \circledcirc = \infty \\ 1 - exp^{-\left((-LN(1-\ell))^{\circledcirc} + (-LN(1-v))^{\circledcirc}\right)^{\frac{1}{\circledcirc}}} & otherwise \end{cases}. \tag{5}$$

where limiting values are $T^{\infty}_A = min$, $T^0_A = T_D$, $T^1_A = T_p$, $S^{\infty}_A = max$, $S^0_A = S_D$, and $S^1_A = S_p$. The t-norm $T^{\circledcirc}_A$ and t-conorm $S^{\circledcirc}_A$ are dual with regard to each other for all $\circledcirc\epsilon[0, \infty]$.

Definition 4 [13]. Let $\sigma = [(\psi^l_\sigma, \mathcal{M}^l_\sigma), (\psi^U_\sigma, \mathcal{M}^U_\sigma)]$, $\sigma_1 = \left[\left(\psi^l_{\sigma_1}, \mathcal{M}^l_{\sigma_2}\right), \left(\psi^U_{\sigma_1}, \mathcal{M}^U_{\sigma_2}\right)\right]$, and $\sigma_2 = \left[\left(\psi^l_{\sigma_1}, \mathcal{M}^l_{\sigma_2}\right), \left(\psi^U_{\sigma_1}, \mathcal{M}^U_{\sigma_2}\right)\right]$ be three IVPyFVs, with $\circledcirc \geq 1$ and $\varsigma \geq 0$. Then, the Aczel-Alsina t-norm and t-conorm operations of IVPyFVs are defined as:

$$\sigma_1 \oplus \sigma_2 = \left(\begin{array}{c} \left[\sqrt{1 - exp^{-\left(\left(-LN\left(1-\psi^{2l}_{\sigma_1}\right)\right)^{\circledcirc} + \left(-LN\left(1-\psi^{2l}_{\sigma_2}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1 - exp^{-\left(\left(-LN\left(1-\psi^{2U}_{\sigma_1}\right)\right)^{\circledcirc} + \left(-LN\left(1-\psi^{2U}_{\sigma_2}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}\right], \\ \left[\sqrt{exp^{-\left(\left(-LN\left(\mathcal{M}^{2l}_{\sigma_1}\right)\right)^{\circledcirc} + \left(-LN\left(\mathcal{M}^{2l}_{\sigma_2}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}, \sqrt{exp^{-\left(\left(-LN\left(\mathcal{M}^{2U}_{\sigma_1}\right)\right)^{\circledcirc} + \left(-LN\left(\mathcal{M}^{2U}_{\sigma_{2U}}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}\right] \end{array}\right), \tag{6}$$

$$\sigma_1 \otimes \sigma_2 = \left(\begin{array}{c} \left[\sqrt{exp^{-\left(\left(-LN\left(\psi^{2l}_{\sigma_1}\right)\right)^{\circledcirc} + \left(-LN\left(\psi^{2l}_{\sigma_2}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{exp^{-\left(\left(-LN\left(\psi^{2U}_{\sigma_1}\right)\right)^{\circledcirc} + \left(-LN\left(\psi^{2U}_{\sigma_2}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}\right], \\ \left[\sqrt{1 - exp^{-\left(\left(-LN\left(1-\mathcal{M}^{2l}_{\sigma_1}\right)\right)^{\circledcirc} + \left(-LN\left(1-\mathcal{M}^{2l}_{\sigma_2}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1 - exp^{-\left(\left(-LN\left(1-\mathcal{M}^{2U}_{\sigma_1}\right)\right)^{\circledcirc} + \left(-LN\left(1-\mathcal{M}^{2U}_{\sigma_2}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}\right] \end{array}\right), \tag{7}$$

$$\varsigma\sigma = \begin{pmatrix} \left[\sqrt{1-exp^{-\left(\varsigma\left(-LN\left(1-\psi_{\sigma}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1-exp^{-\left(\varsigma\left(-LN\left(1-\psi_{\sigma}^{2U}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}\right], \\ \left[\sqrt{exp^{-\left(\varsigma\left(-LN\left(\mathcal{M}_{\sigma}^{2l}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}, \sqrt{exp^{-\left(\varsigma\left(-LN\left(\mathcal{M}_{\sigma}^{2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}\right] \end{pmatrix}, \tag{8}$$

$$\sigma^{\varsigma} = \begin{pmatrix} \left[\sqrt{exp^{-\left(\varsigma\left(-LN\left(\psi_{\sigma}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{exp^{-\left(\varsigma\left(-LN\left(\psi_{\sigma}^{2U}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}\right], \\ \left[\sqrt{1-exp^{-\left(\varsigma\left(-LN\left(1-\mathcal{M}_{\sigma}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1-exp^{-\left(\varsigma\left(-LN\left(1-\mathcal{M}_{\sigma}^{2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}\right] \end{pmatrix}. \tag{9}$$

Definition 5. Let $\sigma_\tau = \left(\left[\psi_{\sigma_\tau}^{l}, \psi_{\sigma_\tau}^{U}\right], \left[\mathcal{M}_{\sigma_\tau}^{l}, \mathcal{M}_{\sigma_\tau}^{U}\right]\right)$ be a collection of IVPyFVs and $\frac{T_1}{\sum_{\tau=1}^{\emptyset} T_\tau}\sigma_\tau = \left(\frac{T_1}{\sum_{\tau=1}^{\emptyset} T_\tau}\sigma_1, \frac{T_1}{\sum_{\tau=1}^{\emptyset} T_\tau}\sigma_2, \ldots, \frac{T_1}{\sum_{\tau=1}^{\emptyset} T_\tau}\sigma_\emptyset\right)^T$ shows a weight vector of $\sigma_\tau(\tau = 1, 2, \ldots, \emptyset)$ in a manner that allows $\sigma_\tau \in [0, 1]$, $\tau = 1, 2, \ldots \emptyset$, and $\sum_{\tau=1}^{\emptyset} \emptyset_\tau = 1$. So, the IVPyFPAAA operator of dimension $\emptyset$ is:

$$IVPyFPAAA(\sigma_1, \sigma_2, .., \sigma_\emptyset) = \begin{pmatrix} \left(1 - \prod_{\tau=1}^{\emptyset}\left(1-\psi_{\sigma_\tau}^{2l}\right)^{\emptyset_\tau}, 1 - \prod_{\tau=1}^{\emptyset}\left(1-\psi_{\sigma_\tau}^{2U}\right)^{\emptyset_\tau}\right), \\ \left(\prod_{\tau=1}^{\emptyset}\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)^{\emptyset_\tau}, \prod_{\tau=1}^{\emptyset}\left(\psi_{\sigma_\tau}^{2U}\right)^{\emptyset_\tau}\right) \end{pmatrix}. \tag{10}$$

3. Interval-value Pythagorean Fuzzy Prioritized Aczel-Alsina Averaging Aggregation Operators

In this section, we present a few IVPyFPAAA operators by means of the Aczel-Alsina operations. Throughout the article, (τ=1,2…,$\emptyset$) will stand for indexing terms.

Definition 6: Let $\sigma_\tau = \left(\left[\psi_{\sigma_\tau}^{l}, \psi_{\sigma_\tau}^{U}\right], \left[\mathcal{M}_{\sigma_\tau}^{l}, \mathcal{M}_{\sigma_\tau}^{U}\right]\right)$ be an accumulation of IVPyFVs. Then, the IVPyFPAAA operator is defined as:

$$IVPyFPAAA(\sigma_1, \sigma_2, \ldots \sigma_\emptyset) = \oplus_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau}\sigma_\tau. \tag{11}$$

Theorem 1: Let $\sigma_\tau = \left(\left[\psi_{\sigma_\tau}^{l}, \psi_{\sigma_\tau}^{U}\right], \left[\mathcal{M}_{\sigma_\tau}^{l}, \mathcal{M}_{\sigma_\tau}^{U}\right]\right)$ be a collection of IVPyFVs. Then, the aggregated value of σ_τ by utilizing the IVPyFPAAA operator is also an IVIFV given by:

$$IVPyFPAAA(\sigma_1, \sigma_2, \dots \sigma_\emptyset) = \left(\begin{array}{c} \left[\sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}, \right. \\ \left. \sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}} \right], \\ \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}, \right. \\ \left. \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}} \right] \end{array} \right). \tag{12}$$

Proof of Theorem 1 is provided Appendix-1.

Theorem 2: If all $\sigma_\tau = \left(\left[\psi_{\sigma_\tau}^l, \psi_{\sigma_\tau}^U\right], \left[\mathcal{M}_{\sigma_\tau}^l, \mathcal{M}_{\sigma_\tau}^U\right]\right) = ([\psi_\sigma^l, \psi_\sigma^U], [\mathcal{M}_\sigma^l, \mathcal{M}_\sigma^l]) = \sigma$, that is $\sigma_\tau = \sigma$ for all τ. Then, $IVPyFPAAA(\sigma_1, \sigma_2, \dots, \sigma_\emptyset) = \sigma$.

Proof of Theorem 2 is provided Appendix-2.

Theorem 3: Let $\sigma_\tau = \left(\left[\psi_{\sigma_\tau}^l, \psi_{\sigma_\tau}^U\right], \left[\mathcal{M}_{\sigma_\tau}^l, \mathcal{M}_{\sigma_\tau}^U\right]\right)$ be a collection of IVPyFVs. Let $\sigma^- = min(\sigma_1, \beta_2, \dots, \sigma_\emptyset)$ and $\sigma^+ = max(\sigma_1, \beta_2, \dots, \sigma_\emptyset)$. Then, $\sigma^- \leq IVPyFPAAA(\sigma_1, \sigma_2, \dots, \sigma_\emptyset) \leq \sigma^+$.

Proof of Theorem 3 is provided Appendix-3.

Theorem 4: Let σ_τ and σ'_τ be two sets of IVPyFVs. If $\sigma_\tau \leq \sigma'_\tau$ for all τ, then:

$$IVPyFPAAA(\sigma_1, \sigma_2, \dots, \sigma_\emptyset) \leq IVPyFPAAA(\sigma'_1, \sigma'_2, \dots, \sigma'_\emptyset). \tag{13}$$

Proof of Theorem 4: Straightforward.

Theorem 5: Let $\sigma_\tau = \left(\left[\psi_{\sigma_\tau}^l, \psi_{\sigma_\tau}^U\right], \left[\mathcal{M}_{\sigma_\tau}^l, \mathcal{M}_{\sigma_\tau}^U\right]\right)$, be a collection of IVPyFVs. Also, $T_{\tau =} \prod_{k=1}^{\tau-1} S\,(\sigma_k)\;(\tau = 2, \dots \emptyset)$, $T_1 = 1$, and $S(\sigma_k)$ is the score of IVPyFVs σ_k. If $\alpha = (\psi_\alpha, \psi_\alpha)$ is IVPyFVs on k, then:

$$IVPyFPAAA(\sigma_1 \oplus \alpha, \sigma_2 \oplus \alpha, \dots, \sigma_\emptyset \oplus \alpha) = IVPyFPAAA(\sigma_1, \sigma_2, \dots, \sigma_\emptyset) \oplus \alpha. \tag{14}$$

Proof of Theorem 5 is provided Appendix-4.

Theorem 6: Let $\sigma_\tau = \left(\left[\psi_{\sigma_\tau}^l, \psi_{\sigma_\tau}^U\right], \left[\mathcal{M}_{\sigma_\tau}^l, \mathcal{M}_{\sigma_\tau}^U\right]\right)$ be a collection of IVPyFVs. Also, $T_\tau = \prod_{k=1}^{\tau-1} S\,(\sigma_k)\;(\tau = 2, \dots \emptyset)$, $T_1 = 1$, and $S(\sigma_k)$ is the score of σ_k. If $\varsigma > 0$, then:

$$IVPyFPAAA(\varsigma\sigma_1, \varsigma\sigma_2, \dots, \varsigma\sigma_\emptyset) = \varsigma IVPyFPAAA(\sigma_1, \sigma_2, \dots, \sigma_\emptyset). \tag{15}$$

Proof of Theorem 6 is provided Appendix-5.

Theorem 7: Let $\sigma_\tau = \left(\left[\psi_{\sigma_\tau}^l, \psi_{\sigma_\tau}^U\right], \left[\mathcal{M}_{\sigma_\tau}^l, \mathcal{M}_{\sigma_\tau}^U\right]\right)$ be a collection of IVPyFVs, and $T_\tau = \prod_{k=1}^{\tau-1} S\,(\sigma_k)\;(\tau = 2, \dots \emptyset)$, $T_1 = 1$, and $S(\sigma_k)$ is the score of IVPyFVs σ_k. If $\varsigma > 0$, $\alpha = ([\psi_\alpha^l, \psi_\alpha^U], [\mathcal{M}_\alpha^l, \mathcal{M}_\alpha^l])$ is IVPyFVs on k. Then:

$$IVPyFPAAA(\varsigma\sigma_1 \oplus \alpha, \varsigma\sigma_2 \oplus \alpha, \dots, \varsigma\sigma_\emptyset \oplus \alpha) = \varsigma \mathrm{IV}PyFPAAA(\sigma_1, \sigma_2, \dots, \sigma_\emptyset) \oplus \alpha. \tag{16}$$

Proof of Theorem 7 is provided Appendix-6.

Theorem 8: Let $\sigma_{\tau=}\left(\left[\psi^l_{\sigma_\tau}, \psi^U_{\sigma_\tau}\right], \left[\mathcal{M}^l_{\sigma_\tau}, \mathcal{M}^U_{\sigma_\tau}\right]\right)$ and $\alpha_{\tau=}\left(\left[\psi^l_{\alpha_\tau}, \psi^U_{\alpha_\tau}\right], \left[\mathcal{M}^l_{\alpha_\tau}, \mathcal{M}^U_{\alpha_\tau}\right]\right)$ be two collections of IVPyFVs, and $T_{\tau=} \prod_{k=1}^{\tau-1} S\ (\sigma_k)\ (\tau = 2, \dots \emptyset)$, $T_1 = 1$, $S(\sigma_k)$ be the score of IVPyFVs σ_k. Then:

$$\begin{gathered} IVPyFPAAA(\sigma_1 \oplus \alpha_1, \sigma_2 \oplus \alpha_2, \dots, \sigma_\emptyset \oplus \alpha_\emptyset) \\ = IVPyFPAAA(\sigma_1, \sigma_2, \dots, \sigma_\emptyset) \oplus\ IVPyFPAAA(\alpha_1, \alpha_2, \dots, \alpha_\emptyset). \end{gathered} \tag{17}$$

Proof of Theorem 8 is provided Appendix-7.

Definition 7: Let $\sigma_\tau = \left(\left[\psi^l_{\sigma_\tau}, \psi^U_{\sigma_\tau}\right], \left[\mathcal{M}^l_{\sigma_\tau}, \mathcal{M}^U_{\sigma_\tau}\right]\right)$ be a collection of IVPyFVs. Then:

$$IVPyFPAAG(\sigma_1, \sigma_2, \dots \sigma_\emptyset) = \otimes_{\tau=1}^{\emptyset} \sigma_\tau^{\frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau}}. \tag{18}$$

The function IVPyFPAAG is called the IVPyFPAAG operator, where $T_{\tau=} \prod_{k=1}^{\tau-1} S\ (\sigma_k)\ (\tau = 2, \dots \emptyset)$, $T_1 = 1$, and $S(\sigma_k)$ is the score of IVPyFVs σ_k.

Theorem 9: Let $\sigma_\tau = \left(\left[\psi^l_{\sigma_\tau}, \psi^U_{\sigma_\tau}\right], \left[\mathcal{M}^l_{\sigma_\tau}, \mathcal{M}^U_{\sigma_\tau}\right]\right)$ be a collection of IVPyFVs. Then, an aggregated value by using the IVPyFPAAG operator is also IVPyFVs given by:

$$IVPyFPAAG(\sigma_1, \sigma_2, \dots \sigma_\emptyset) = \left(\begin{gathered} \left[\begin{gathered} \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau}\left(-LN\left(\mathcal{M}^{2l}_{\sigma_\tau}\right)\right)^{\text{\textcircled{0}}}\right)^{\frac{1}{\text{\textcircled{0}}}}}}, \\ \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau}\left(-LN\left(\mathcal{M}^{2U}_{\sigma_\tau}\right)\right)^{\text{\textcircled{0}}}\right)^{\frac{1}{\text{\textcircled{0}}}}}} \end{gathered} \right], \\ \left[\begin{gathered} \sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau}\left(-LN\left(1-\psi^{2l}_{\sigma_\tau}\right)\right)^{\text{\textcircled{0}}}\right)^{\frac{1}{\text{\textcircled{0}}}}}}, \\ \sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau}\left(-LN\left(1-\psi^{2U}_{\sigma_\tau}\right)\right)^{\text{\textcircled{0}}}\right)^{1/\text{\textcircled{0}}}}} \end{gathered} \right] \end{gathered} \right). \tag{19}$$

Proof of Theorem 9 is provided Appendix-8.

The IVPyFPAAG operator is likely to satisfy the properties listed below, which are provided in Theorems 10-12.

Theorem 10: If all $\sigma_\tau = \left(\left[\psi^l_{\sigma_\tau}, \psi^U_{\sigma_\tau}\right], \left[\mathcal{M}^l_{\sigma_\tau}, \mathcal{M}^U_{\sigma_\tau}\right]\right)$ are a collection of IVPyFVs, where $T_{\tau=} \prod_{k=1}^{\tau-1} S\ (\sigma_k)\ (\tau = 2, \dots \emptyset)$, $T_1 = 1$, and $S(\sigma_k)$ is the score of IVPyFVs (σ_k). If all $\sigma_\tau = \sigma$ for all τ, then $IVPyFPAAG(\sigma_1, \sigma_2, \dots, \sigma_\emptyset) = \sigma$.

Proof of Theorem 10 is provided Appendix-9.

Theorem 11: Let $\sigma_\tau = \left(\left[\psi^l_{\sigma_\tau}, \psi^U_{\sigma_\tau}\right], \left[\mathcal{M}^l_{\sigma_\tau}, \mathcal{M}^U_{\sigma_\tau}\right]\right)$ be a collection of IVPyFVs. Let $\sigma^- = min(\sigma_1, \beta_2, \dots, \sigma_\emptyset)$ and $\sigma^+ = max(\sigma_1, \beta_2, \dots, \sigma_\emptyset)$. Then, $\sigma^- \leq IVPyFPAAG(\sigma_1, \sigma_2, \dots, \sigma_\emptyset) \leq \sigma^+$.

Proof of Theorem 11 is provided Appendix-10.

Theorem 12: Let $\sigma_\tau = \left(\left[\psi^l_{\sigma_\tau}, \psi^U_{\sigma_\tau}\right], \left[\mathcal{M}^l_{\sigma_\tau}, \mathcal{M}^U_{\sigma_\tau}\right]\right)$, be a collection of IVPyFVs, where $T_{\tau=} \prod_{k=1}^{\tau-1} S\ (\sigma_k)\ (\tau = 2, \dots \emptyset)$, $T_1 = 1$, and $S(\sigma_k)$ is the score of IVPyFVs(σ_k). If $\alpha = (\psi_\alpha, \psi_\alpha)$ is IVPyFVs on k, then:

$$IVPyFPAAG(\sigma_1 \otimes \alpha, \sigma_2 \otimes \alpha, \dots, \sigma_\emptyset \otimes \alpha) = IVPyFPAAG(\sigma_1, \sigma_2, \dots, \sigma_\emptyset) \otimes \alpha. \tag{20}$$

Proof of Theorem 12: Proof Theorem 5 can be used to prove Theorem 12, too.

Theorem 13: Let $\sigma_\tau = \left([\psi^l_{\sigma_\tau}, \psi^U_{\sigma_\tau}], [\mathcal{M}^l_{\sigma_\tau}, \mathcal{M}^U_{\sigma_\tau}]\right)$, be a collection of IVPyFVs, where $T_{\tau=} \prod_{k=1}^{\tau-1} S\,(\sigma_k)\,(\tau = 2, \dots \emptyset)$, $T_1 = 1$, and $S(\sigma_k)$ is the score of IVPyFVs(σ_k). If $r > 0$, then:

$$IVPyFPAAG\left(\sigma_1^r, \sigma_2^r, \dots, \sigma_\emptyset^r\right) = IVPyFPAAG(\sigma_1, \sigma_2, \dots, \sigma_\emptyset)^r. \tag{21}$$

Proof of Theorem 13: Proof of Theorem 6 can be used to prove Theorem 13, too.

Theorem 14: Let $\sigma_\tau = \left([\psi^l_{\sigma_\tau}, \psi^U_{\sigma_\tau}], [\mathcal{M}^l_{\sigma_\tau}, \mathcal{M}^U_{\sigma_\tau}]\right)$, be a collection of IVPyFVs, where $T_{\tau=} \prod_{k=1}^{\tau-1} S\,(\sigma_k)\,(\tau = 2, \dots \emptyset)$, $T_1 = 1$, and $S(\sigma_k)$ is the score of IVPyFVs(σ_k). If $r > 0$, $\alpha = (\psi_\alpha, \psi_\alpha)$ is IVPyFVs on k. Then:

$$IVPyFPAAG\left(\sigma_1^r \otimes \alpha, \sigma_2^r \otimes \alpha, \dots \sigma_\emptyset^r \otimes \alpha\right) = IVPyFPAAG(\sigma_1, \sigma_2, \dots, \sigma_\emptyset)^r \otimes \alpha. \tag{22}$$

Proof of Theorem 14: Proof of Theorem 7 can be used to prove Theorem 14, too.

Theorem 15: Let $\sigma_{\tau=}\left([\psi^l_{\sigma_\tau}, \psi^U_{\sigma_\tau}], [\mathcal{M}^l_{\sigma_\tau}, \mathcal{M}^U_{\sigma_\tau}]\right)$ and $\alpha_{\tau=}\left([\psi^l_{\sigma_\tau}, \psi^U_{\sigma_\tau}], [\mathcal{M}^l_{\sigma_\tau}, \mathcal{M}^U_{\sigma_\tau}]\right)$ be two collections of IVPyFVs. Also, $T_{\tau=} \prod_{k=1}^{\tau-1} S\,(\sigma_k)\,(\tau = 2, \dots \emptyset)$, $T_1 = 1$, and $S(\sigma_k)$ be the score of IVIFVsσ_k. Then:

$$\begin{aligned} &IVPyFPAAG(\sigma_1 \otimes \alpha_1, \sigma_2 \otimes \alpha_2, \dots, \sigma_\emptyset \otimes \alpha_\emptyset) \\ &= IVPyFPAAG(\sigma_1, \sigma_2, \dots, \sigma_\emptyset) \otimes \ IVIFPAAG(\alpha_1, \alpha_2, \dots, \alpha_\emptyset). \end{aligned} \tag{23}$$

Proof of Theorem 15: Proof of Theorem 8 can be used to prove Theorem 15, too.

4. MAGDM Methods by using Investigated Operators Based on IVPyFVs

We will take the following actions to put the recommended aggregation operators into practice for a MAGDM methodology in the IVPyF environment. For example, let $k = \{k_1, k_2, \dots k_m\}$ be the set of alternatives. Assume that the set of criteria is $A = \{a_1, a_2, \dots, a_n\}$. Using a linear ordering such that $a_1 > a_2 > a_3, \dots, a_n$, we have $A = \{a_1, a_2, \dots, a_n\}$. This ordering suggests that if a_τ are criteria, then $\tau < i$, implies that a_τ is more important than a_i. Assume that $E = \{e_1, e_2, \dots, e_p\}$ represents the group of decision-makers. The linear hierarchy $e_1 > e_2 >, \dots, > e_p$ implies a hierarchy between them, with exp_q and k_i serving as decision-makers. $\partial < \tau$ Implies that exp_q is more important than k_i. Considering IVPyFVs, $\left(\psi^q_{i\tau}, \psi^q_{i\tau}\right)$ indicates the degree range in which the alternative k_i satisfies the attribute c_τ expressed be the decision maker exp_q, such that $\left(\psi^q_{i\tau}, \psi^q_{i\tau}\right) \subset (0,1)$, $\left(\psi^q_{i\tau} + \psi^q_{i\tau}\right) \leq 1$ $(i = 1, 2, \dots m;\ \tau = 1, 2, \dots n)$.

If all the attributes $a_\tau(\tau = 1, 2, \dots, n)$ are of the same type, then the attribute values do not need normalization. Otherwise, we normalize the decision-maker matrix $K^q = \left(K^q_{i\tau}\right)_{mxn}$ into $R^q = \left(r^q_{i\tau}\right)_{mxn}$, where:

$$r^q_{i\tau} = \begin{cases} k^q_{i\tau}, \text{for the benefit attribute } a_\tau \\ \ k^q_{i\tau}, \text{for the cost attribute } a_\tau \end{cases}. \tag{24}$$

Step 1: Calculate the values of $T^q_{i\tau}$ as:

$$T_{\tau=} \prod_{k=1}^{\tau-1} S\left(r^q_{i\tau}\right)(q = 2, \dots p), T^1_{i\tau} = 1. \tag{25}$$

Step 2: To aggregate all the individual IVPyF decision matrices $R^q = \left(r_{i\tau}^q\right)_{mxn} (q = 2, \dots p)$ into the collective IVPyF decision matrix $R = (r_{i\tau})_{mxn}$ utilize the IVPyFPAAA operator:

$$r_{i\tau} = IVPyFPAAA\left(\sigma_{i\tau}{}^1, \sigma_{i\tau}{}^2, \dots, \sigma_{i\tau}{}^p{}_{\emptyset}\right)$$
$$= \left(\begin{array}{c}\left[\sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau}^{(p)}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{@}\right)^{\frac{1}{@}}}}, \sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau}^{(p)}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{@}\right)^{\frac{1}{@}}}}\right], \\ \left[\sqrt{expxp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau}^{(p)}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{@}\right)^{1/@}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau}^{(p)}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{@}\right)^{1/@}}}\right]\end{array}\right), \tag{26}$$

or the IVPyFPAAG operator:

$$r_{i\tau} = IVPyFPAAG\left(\sigma_{i\tau}{}^1, \sigma_{i\tau}{}^2, \dots \sigma_{i\tau}{}^p{}_{\emptyset}\right)$$
$$= \left(\begin{array}{c}\left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau}^{(p)}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{@}\right)^{\frac{1}{@}}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau}^{(p)}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{@}\right)^{\frac{1}{@}}}}\right], \\ \left[\sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau}^{(p)}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{@}\right)^{\frac{1}{@}}}}, \sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau}^{(p)}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{@}\right)^{1/@}}}\right]\end{array}\right). \tag{27}$$

Step 3: Calculate the values of $T_{i\tau}$ as:

$$T_{\tau=} \prod_{k=1}^{\tau-1} S\left(r_{i\tau}^q\right)(q = 2, \dots p), T_{i\tau}^1 = 1. \tag{28}$$

Step 4: Aggregate IVPyFVs as:

$$IVPyFPAAA(\sigma_1, \sigma_2, \dots \sigma_\emptyset) = \left(\begin{array}{c}\left[\begin{array}{c}\sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{@}\right)^{\frac{1}{@}}}}, \\ \sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{@}\right)^{\frac{1}{@}}}}\end{array}\right], \\ \left[\begin{array}{c}\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{@}\right)^{\frac{1}{@}}}}, \\ \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{@}\right)^{1/@}}}\end{array}\right]\end{array}\right), \tag{29}$$

or as:

$$IVPyFPAAG(\sigma_1, \sigma_2, \dots \sigma_\emptyset) = \left(\begin{array}{c} \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}} \right], \\ \left[\sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}, \sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}} \right] \end{array} \right). \tag{30}$$

Step 5: Rank all the alternatives by the score function as:

$$ISco(\sigma_i) = \frac{\psi_{\sigma_1}^{l} + \psi_{\sigma_1}^{U} - \mathcal{M}_{\sigma_1}^{l} - \mathcal{M}_{\sigma_1}^{U}}{2} \quad i = 1, 2, \dots m, \tag{31}$$

where the bigger the value of $S(r_i)$ means the larger the overall IVPyFVs (r_i) and thus the alternative $k_i (i = 1, 2, \dots m)$.

5. Practical Example

When choosing how to distribute its products, a manufacturing company gives priority to environmentally friendly transportation options. Based on a number of variables, the company is weighing four distinct transportation options in order to make an informed choice. Below is a brief discussion of these options:

i. *Electric vehicles* (K_1) – By running on electricity, electric trucks lessen their dependency on fossil fuels. Since they do not have internal combustion engines but rather electric motors, they emit no tailpipe emissions. When paired with renewable energy sources for charging, this leads to notably reduced greenhouse gas emissions when compared to traditional diesel trucks.

ii. *Biofuel-powered airplanes* (K_2) – Renewable biofuels made from waste materials, plants, or algae are used in biofuel-powered aircraft. These biofuels are intended to minimize carbon emissions and lessen dependency on traditional aviation fuels derived from fossil fuels. The aviation sector is still experimenting with biofuels, though, because of their inconsistent availability and scalability.

iii. *Rail transportation* (K_3) – Rail transportation is renowned for its low environmental impact and energy efficiency. An environmentally friendly method of transporting goods over land is through diesel-powered locomotives or electrically powered trains running on electrified lines. Trains are effective for long-distance transportation because they can carry heavy loads and emit fewer emissions per ton-mile.

iv. *Hybrid cargo ships* (K_4) – In order to lower fuel consumption and pollution, hybrid cargo ships blend traditional fuel with alternative energy sources like electricity or wind power. In addition to conventional engines, these ships frequently make use of technologies like sails, solar panels, or batteries. They use less harmful energy sources, but they still use some conventional fuels, so emissions are decreased but not completely eliminated.

v. *Solar-powered delivery drones* (K_5) – Delivery drones that run on solar energy consume less fossil fuel and emit fewer greenhouse gases when they propel themselves. They can function independently and are appropriate for small package deliveries. In cities, these drones provide an environmentally friendly last-mile delivery option.

The following standards are used by the company to assess the available transportation options:

i. *Environmental effect* (a_1) – This criterion evaluates each mode of transportation's overall environmental impact, taking into account things like pollution, carbon emissions, and ecological footprint.
ii. *Cost efficiency* (a_2) – This criterion determines the total cost of utilizing a mode of transportation, accounting for initial, continuous, and maintenance costs.
iii. *Delivery time* (a_3) – This criterion assesses the speed and dependability with which each mode of transportation can deliver goods to their designated destination.
iv. *Capacity* (a_4) – This criterion establishes the maximum weight or volume that each mode of transportation can safely handle.

The candidates $K_i(i = 1, 2, \dots 5)$ were assessed by three decision-makers in relation to the attributes $A_\tau(\tau = 1, 2, \dots, 5)$ and three IVPyF decision matrices were created $D^q = \left(d_{i\tau}^q\right)_{5x4}(q = 1, 2, 3)$ and given in Tables 1-3. Normalization of the decision matrices was not required because all of the attributes $K_i(i = 1, 2, \dots 5)$ were of the same type.

Table 1
Interval-value Pythagorean fuzzy decision matrix e_1

	a_1		a_2		a_3		a_4	
k_1	[0.44,0.48]	[0.34,0.44]	[0.35,0.45]	[0.34,0.42]	[0.33,0.54]	[0.36,0.44]	[0.34,0.53]	[0.35,0.45]
k_2	[0.38,0.54]	[0.35,0.48]	[0.44,0.48]	[0.31,0.44]	[0.45,0.53]	[0.43,0.46]	[0.33,0.48]	[0.33,0.44]
k_3	[0.45,0.52]	[0.33,0.45]	[0.42,0.55]	[0.37,0.53]	[0.37,0.55]	[0.43,0.49]	[0.37,0.51]	[0.36,0.45]
k_4	[0.45,0.54]	[0.42,0.47]	[0.36,0.59]	[0.39,0.56]	[0.35,0.56]	[0.34,0.52]	[0.39,0.55]	[0.35,0.47]
k_5	[0.37,0.49]	[0.37,0.42]	[0.35,0.58]	[0.37,0.45]	[0.44,0.59]	[0.45,0.54]	[0.36,0.56]	[0.38,0.41]

Table 2
Interval-value Pythagorean fuzzy decision matrix e_2

	a_1		a_2		a_3		a_4	
k_1	[0.36,0.53]	[0.37,0.41]	[0.34,0.52]	[0.34,0.45]	[0.34,0.54]	[0.34,0.45]	[0.45,0.48]	[0.34,0.52]
k_2	[0.45,0.53]	[0.44,0.45]	[0.42,0.52]	[0.42,0.48]	[0.45,0.56]	[0.33,0.48]	[0.42,0.53]	[0.43,0.45]
k_3	[0.42,0.58]	[0.38,0.53]	[0.44,0.59]	[0.44,0.53]	[0.38,0.51]	[0.26,0.49]	[0.48,0.57]	[0.46,0.48]
k_4	[0.42,0.56]	[0.26,0.54]	[0.48,0.59]	[0.46,0.55]	[0.48,0.56]	[0.38,0.52]	[0.49,0.54]	[0.35,0.49]
k_5	[0.39,0.43]	[0.22,0.39]	[0.46,0.59]	[0.42,0.57]	[0.44,0.55]	[0.37,0.54]	[0.38,0.56]	[0.32,0.45]

Table 3
Interval-value Pythagorean fuzzy decision matrix e_3

	a_1		a_2		a_3		a_4	
k_1	[0.33,0.47]	[0.34,0.43]	[0.45,0.58]	[0.45,0.54]	[0.43,0.56]	[0.45,0.49]	[0.45,0.56]	[0.38,0.56]
k_2	[0.45,0.48]	[0.31,0.46]	[0.45,0.57]	[0.42,0.52]	[0.38,0.53]	[0.43,0.47]	[0.43,0.51]	[0.36,0.53]
k_3	[0.43,0.45]	[0.35,0.44]	[0.37,0.48]	[0.38,0.45]	[0.41,0.55]	[0.42,0.45]	[0.44,0.58]	[0.39,0.57]
k_4	[0.36,0.41]	[0.29,0.34]	[0.35,0.57]	[0.39,0.52]	[0.46,0.57]	[0.38,0.46]	[0.43,0.52]	[0.34,0.52]
k_5	[0.38,0.59]	[0.43,0.52]	[0.38,0.59]	[0.35,0.51]	[0.45,0.53]	[0.36,0.47]	[0.45,0.56]	[0.42,0.53]

When utilizing the IVPyFPAAA operator, the main steps are as follows since the attribute values do not need to be normalized after changing type:

Step 1: The values of $T_{i\tau}^1, T_{i\tau}^2$, and $T_{i\tau}^3$ are:

$$T_{i\tau}^1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},\ T_{i\tau}^2 = \begin{pmatrix} 0.06 & 0.02 & 0.04 & 0.04 \\ 0.04 & 0.07 & 0.04 & 0.02 \\ 0.08 & 0.03 & 0.01 & 0.03 \\ 0.05 & 0.01 & 0.03 & 0.06 \\ 0.03 & 0.06 & 0.02 & 0.07 \end{pmatrix}, \text{and } T_{i\tau}^3 = \begin{pmatrix} 0.003 & 0.001 & 0.002 & 0.001 \\ 0.002 & 0.001 & 0.004 & 0.001 \\ 0.004 & 0.001 & 0.000 & 0.002 \\ 0.003 & 0.000 & 0.002 & 0.005 \\ 0.002 & 0.002 & 0.001 & 0.005 \end{pmatrix}.$$

Step 2: By utilizing the IVPyFPAAA operator, each distinct IVPyF decision matrix $R^q = \left(r_{i\tau}^q\right)_{4x5}(q = 1,2,3)$ was aggregated into the collective IVPyF decision matrix $R = (r_{i\tau})_{5x4}$, as shown in Table 4.

Table 4
Collective interval-value Pythagorean fuzzy decision matrix R by the IVPyFPAAA operator

	$\boldsymbol{a_1}$		$\boldsymbol{a_2}$	
$\boldsymbol{k_1}$	[0.4359,0.4829]	[0.3416,0.4383]	[0.3499,0.4514]	[0.3401,0.4205]
$\boldsymbol{k_2}$	[0.3833,0.5395]	[0.3531,0.4787]	[0.4388,0.4828]	[0.3161,0.4425]
$\boldsymbol{k_3}$	[0.4478,0.5247]	[0.3335,0.4555]	[0.4206,0.5512]	[0.3719,0.5299]
$\boldsymbol{k_4}$	[0.4473,0.5407]	[0.4105,0.4726]	[0.3609,0.5900]	[0.3904,0.5599]
$\boldsymbol{k_5}$	[0.3707,0.4886]	[0.3642,0.4192]	[0.3575,0.5806]	[0.3726,0.4561]
	a_3		a_4	
$\boldsymbol{k_1}$	[0.3306,0.5400]	[0.3594,0.4404]	[0.3447,0.5284]	[0.3497,0.4523]
$\boldsymbol{k_2}$	[0.4498,0.5313]	[0.4253,0.4608]	[0.3320,0.4810]	[0.3316,0.4402]
$\boldsymbol{k_3}$	[0.3701,0.5497]	[0.4285,0.4900]	[0.3743,0.5122]	[0.3628,0.4511]
$\boldsymbol{k_4}$	[0.3542,0.5600]	[0.3410,0.5199]	[0.3964,0.5494]	[0.3500,0.4712]
$\boldsymbol{k_5}$	[0.4400,0.5891]	[0.4479,0.5399]	[0.3618,0.5600]	[0.3762,0.4128]

Step 3: The values of $T_{i\tau}$ were computed as:

$$T_{i\tau} = \begin{pmatrix} 1 & 0.123 & 0.001 & 0.000 \\ 1 & 0.357 & 0.003 & 0.000 \\ 1 & 0.115 & 0.002 & 0.000 \\ 1 & 0.142 & 0.000 & 0.000 \\ 1 & 0.477 & 0.002 & 0.000 \end{pmatrix}.$$

Step 4: The IVPyFPAAA operator was used to combine all of the desire values $r_{i\tau}(i = 1, 2, \dots, 5)$ on the *i*-th line of R to obtain the overall preference values as r_1=[[0.4276, 0.4797], [0.3414, 0.4363]], r_2=[[0.3990, 0.5256], [0.3431, 0.4689]], r_3=[[0.4450, 0.5276], [0.3375, 0.4627]], r_4=[[0.4378, 0.5473], [0.4079, 0.4827]], and r_5=[[0.3666, 0.5216], [0.3670, 0.4309]].

Step 5: Each score was determined as s_1=0.0530, s_2=0.0489, s_3=0.0742, s_4=0.0459, and s_5=0.0430. Since $S_3 > S_1 > S_2 > S_4 > S_5$, we have $k_3 > k_1 > k_2 > k_4 > k_5$.

When utilizing the IVPyFPAAG operator, the main steps are as follows:

Step 1: Already done.

Step 2: By utilizing the IVPyFPAAG operator, each distinct IVPyF decision matrix $R^q = \left(r_{i\tau}^q\right)_{4x5}(q = 1,2,3)$ was aggregated into the collective IVPyF decision matrix $R = (r_{i\tau})_{5x4}$, as shown in Table 5.

Step 3: The values of $T_{i\tau}$ were computed as:

$$T_{i\tau} = \begin{pmatrix} 1 & 0.0565 & 0.0009 & 0.000 \\ 1 & 0.0414 & 0.0026 & 0.0001 \\ 1 & 0.0774 & 0.0024 & 0.000 \\ 1 & 0.0481 & 0.0003 & 0.000 \\ 1 & 0.0327 & 0.0019 & 0.000 \end{pmatrix}.$$

Table 5
Collective interval-value Pythagorean fuzzy decision matrix R by the IVPyFPAAG operator

	a_1		a_2	
k_1	[0.4349,0.4826]	[0.3417,0.4384]	[0.3499,0.4511]	[0.3401,0.4206]
k_2	[0.3827,0.5395]	[0.3541,0.4788]	[0.4387,0.4825]	[0.3186,0.4428]
k_3	[0.4476,0.5240]	[0.3341,0.4567]	[0.4205,0.5511]	[0.3723,0.5299]
k_4	[0.4468,0.5405]	[0.4141,0.4734]	[0.3606,0.5900]	[0.3905,0.5599]
k_5	[0.3706,0.4882]	[0.3666,0.4194]	[0.3555,0.5806]	[0.3730,0.4583]
	a_3		a_4	
k_1	[0.3305,0.5400]	[0.3595,0.4405]	[0.3434,0.5282]	[0.3497,0.4528]
k_2	[0.4497,0.5312]	[0.4265,0.4609]	[0.3315,0.4809]	[0.3322,0.4402]
k_3	[0.3701,0.5497]	[0.4291,0.4900]	[0.3731,0.5119]	[0.3638,0.4512]
k_4	[0.3529,0.5600]	[0.3411,0.5199]	[0.3949,0.5493]	[0.3500,0.4713]
k_5	[0.4400,0.5890]	[0.4483,0.5400]	[0.3616,0.5600]	[0.3769,0.4133]

Step 4: The IVPyFPAAG operator was used to combine all of the desire values $r_{i\tau}(i = 1, 2, \ldots, 5)$ on the *i*-th line of R to obtain the overall preference values as r_1=[[0.4298, 0.4809], [0.3416, 0.4375]], r_2=[[0.3849, 0.5371], [0.3530, 0.4774]], r_3=[[0.4454, 0.5259], [0.3373, 0.4627]], r_4=[[0.4424, 0.5427], [0.4130, 0.4779]], and r_5=[[0.3702, 0.4911], [0.3670, 0.4210]].

Step 5: Each score was determined as s_1=0.0530, s_2=0.0489, s_3=0.0742, s_4=0.0459, and s_5=0.0430. Since $S_3 > S_1 > S_4 > S_2 > S_5$, we have $k_3 > k_1 > k_4 > k_2 > k_5$. Therefore, k_3 is the best alternative.

6. Comparative Study

This section compares the combined outcomes of using the IVPyFPAAG and IVPyFPAAA operators for IVPyFVs, along with a variety of other aggregation operators; i.e. the IFPAAA [30], PyFPAAA [31], and IVIFFWA operator [32]. To fully evaluate these operators, we applied them to the solution of the previously discussed problem. Table 6 below compiles the results.

Table 6
Results of the comparative analysis

Operator	Ranking
$IVIFPAAA$	$\mathrm{k}_3 > \mathrm{k}_1 > \mathrm{k}_4 > \mathrm{k}_2 > \mathrm{k}_5$
$IVIFPAAG$	$\mathrm{k}_3 > \mathrm{k}_1 > \mathrm{k}_4 > \mathrm{k}_2 > \mathrm{k}_5$
$IFPAAA$ [30]	$\mathrm{k}_3 > \mathrm{k}_1 > \mathrm{k}_4 > \mathrm{k}_2 > \mathrm{k}_5$
$IFPAAG$ [30]	$\mathrm{k}_2 > \mathrm{k}_1 > \mathrm{k}_3 > \mathrm{k}_4 > \mathrm{k}_5$
$PyFPAAA$ [31]	$\mathrm{k}_3 > \mathrm{k}_1 > \mathrm{k}_4 > \mathrm{k}_2 > \mathrm{k}_5$
$PyFPAAG$ [31]	$\mathrm{k}_2 > \mathrm{k}_1 > \mathrm{k}_3 > \mathrm{k}_4 > \mathrm{k}_5$
$IVIFFWA$ [32]	$\mathrm{k}_3 > \mathrm{k}_1 > \mathrm{k}_4 > \mathrm{k}_2 > \mathrm{k}_5$
$IVIFFWA$ [33]	$\mathrm{k}_3 > \mathrm{k}_1 > \mathrm{k}_4 > \mathrm{k}_2 > \mathrm{k}_5$

Operators IVPyFPAAG and IVPyFPAAA consistently determine that k_3 is the best option. In comparison, the aggregation operators presented by the IFPAAA [30], PyFPAAA[31], and IVIFFWA operator [32] also produce findings that are comparable and support k_3 as the best option. However, our proposed operators are highly effective because they include attribute prioritization, which increases decision precision. The results from the IVPyFPAAG and IVPyFPAAA operators show a clear and stable order thus demonstrating their adaptability in handling complex decision scenarios. Figure 1 provides a graphic representation of this comparative analysis, highlighting the alignment and differences in results between the various aggregation techniques that were employed.

Spectrum of Engineering and Management Sciences
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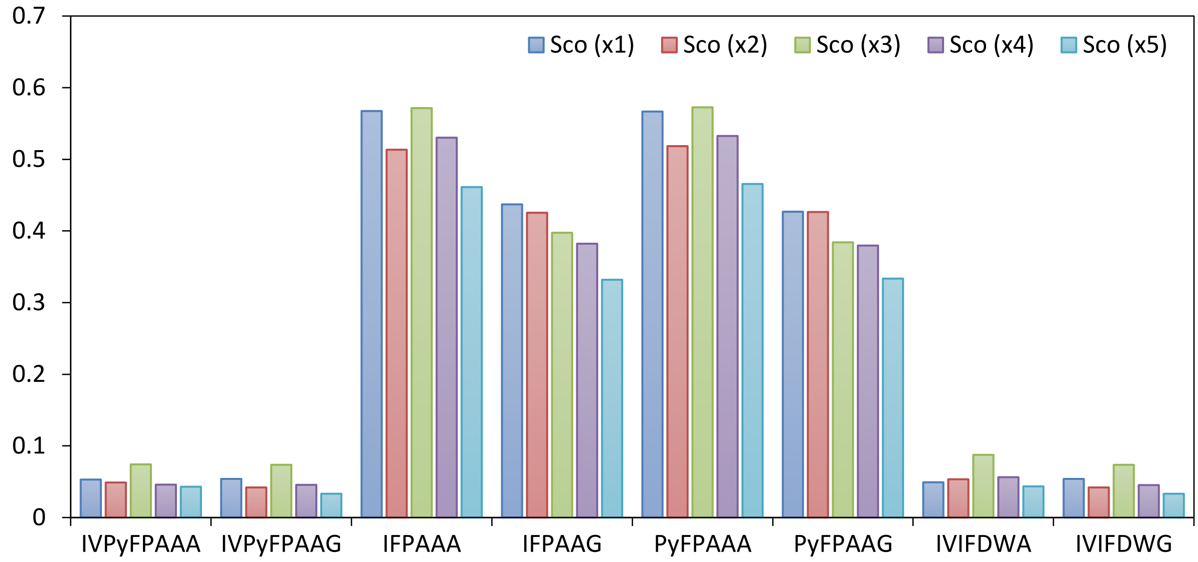

Fig. 1. Results of the comparative study

7. Conclusions

The environmental pollution that comes from transportation activities poses serious challenges to the sustainable development of cities worldwide. Shared mobility presents a workable solution to lessen the adverse effects of urban transportation. For PyFVs, we investigated Aczel-Alsina AOs and put forth two new operators; i.e. IVPyFPAAG and IVPyFPAAA. By incorporating prioritization, these operators address the requirement that real-world decision-making scenarios take into account the varying degrees of importance among attributes and decision-makers. We analyzed the properties of the IVPyFPAAG and IVPyFPAAA operators and illustrated their applicability with a numerical example concerning the MAGDM approach. This approach emphasizes how crucial it is to rank the criteria and decision-makers in decision-making processes. When we compared our findings with those of other aggregation operators that included prioritization, we discovered consistent and similar results. In the future, the study will be expanded to complex Pythagorean fuzzy sets, complex q-rung orthopair fuzzy sets, and complex interval-valued Pythagorean fuzzy sets.

Appendix-1: Proof of Theorem 1

Theorem 1 can be demonstrated in the following way using the mathematical induction approach.

For $\emptyset = 2$, using Aczel-Alsina operations of IVPyFVs, we obtain:

$$\frac{T_1}{\sum_{\tau=1}^{\emptyset} T_\tau}\sigma_1 = \left(\begin{array}{c} \left[\sqrt{1 - exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{\emptyset} T_\tau}\left(-LN\left(1-\psi_{\sigma_1}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1 - exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{\emptyset} T_\tau}\left(-LN\left(1-\psi_{\sigma_1}^{2U}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}} \right], \\ \left[\sqrt{exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{\emptyset} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_1}^{2l}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}, \sqrt{exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{\emptyset} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_1}^{2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}} \right] \end{array} \right)$$

$$\frac{T_2}{\sum_{\tau=1}^{\emptyset} T_\tau}\sigma_2 = \left(\begin{array}{c}\left[\sqrt{1-exp^{-\left(\frac{T_2}{\sum_{\tau=1}^{\emptyset} T_\tau}\left(-LN\left(1-\psi_{\sigma_2}^{2l}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}, \sqrt{1-exp^{-\left(\frac{T_2}{\sum_{\tau=1}^{\emptyset} T_\tau}\left(-LN\left(1-\psi_{\sigma_2}^{2U}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}\right], \\ \left[\sqrt{exp^{-\left(\frac{T_2}{\sum_{\tau=1}^{\emptyset} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_2}^{2l}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}}, \sqrt{exp^{-\left(\frac{T_2}{\sum_{\tau=1}^{\emptyset} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_2}^{2U}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}}\right]\end{array}\right)$$

$$IVPyFPAAA(\sigma_1, \sigma_2) = \frac{T_1}{\sum_{\tau=1}^{2} T_\tau}\sigma_1 \oplus \frac{T_2}{\sum_{\tau=1}^{2} T_\tau}\sigma_2$$

$$= \left(\begin{array}{c}\left[\sqrt{1-exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(1-\psi_{\sigma_1}^{2l}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}, \sqrt{1-exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(1-\psi_{\sigma_1}^{2U}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}\right], \\ \left[\sqrt{exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_1}^{2l}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}}, \sqrt{exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_1}^{2U}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}}\right]\end{array}\right) \oplus$$

$$\left(\begin{array}{c}\left[\sqrt{1-exp^{-\left(\frac{T_2}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(1-\psi_{\sigma_2}^{2l}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}, \sqrt{1-exp^{-\left(\frac{T_2}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(1-\psi_{\sigma_2}^{2U}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}\right], \\ \left[\sqrt{exp^{-\left(\frac{T_2}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_2}^{2l}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}}, \sqrt{exp^{-\left(\frac{T_2}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_2}^{2U}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}}\right]\end{array}\right)$$

$$= \left(\begin{array}{c}\left[\sqrt{1-exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(1-\psi_{\sigma_1}^{2l}\right)\right)^{\mathbb{0}}+\frac{T_2}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(1-\psi_{\sigma_2}^{2l}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}, \sqrt{1-exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(1-\psi_{\sigma_1}^{2U}\right)\right)^{\mathbb{0}}+\frac{T_2}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(1-\psi_{\sigma_2}^{2U}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}\right], \\ \left[\sqrt{exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(\mathcal{M}_{1}^{2l}\right)\right)^{\mathbb{0}}+\frac{T_2}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_2}^{2l}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}}, \sqrt{exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_1}^{2U}\right)\right)^{\mathbb{0}}+\frac{T_2}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_2}^{2U}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}}\right]\end{array}\right)$$

$$= \left(\begin{array}{c}\left[\sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}, \sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}\right], \\ \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}}\right]\end{array}\right)$$

Hence, Eq. (12) is true for $\emptyset = 2$.

Assume that Eq. (12) is true for $\emptyset = k$. Then, we have:

$$IVPyFPAAA(\sigma_1, \sigma_2, \dots \sigma_k) = \oplus_{\tau=1}^{k} \frac{T_\tau}{\sum_{\tau=1}^{k} T_\tau}(\sigma_\tau)$$

$$= \left(\begin{array}{c}\left[\sqrt{1-exp^{-\left(\sum_{\tau=1}^{k}\frac{T_\tau}{\sum_{\tau=1}^{k} T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}, \sqrt{1-exp^{-\left(\sum_{\tau=1}^{k}\frac{T_\tau}{\sum_{\tau=1}^{k} T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}\right] \\ \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{k}\frac{T_\tau}{\sum_{\tau=1}^{k} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{k}\frac{T_\tau}{\sum_{\tau=1}^{k} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}}\right]\end{array}\right)$$

Now for $\emptyset = k + 1$, we get:

$$IVPyFPAAA(\sigma_1, \sigma_2, \ldots \sigma_{k+1}) = \oplus_{\tau=1}^{k} \frac{T_\tau}{\sum_{\tau=1}^{k+1} T_\tau}(\sigma_\tau) \oplus \frac{T_{k+1}}{\sum_{\tau=1}^{k+1} T_\tau}(\sigma_{k+1})$$

$$= \left(\begin{array}{c} \left[\sqrt{1 - exp^{-\left(\sum_{\tau=1}^{k} \frac{T_\tau}{\sum_{\tau=1}^{k+1} T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}}, \sqrt{1 - exp^{-\left(\sum_{\tau=1}^{k} \frac{T_\tau}{\sum_{\tau=1}^{k+1} T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}} \right], \\ \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{k} \frac{T_\tau}{\sum_{\tau=1}^{k+1} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{k} \frac{T_\tau}{\sum_{\tau=1}^{k+1} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}} \right] \end{array} \right) \oplus$$

$$\left(\begin{array}{c} \left[\sqrt{1 - exp^{-\left(\frac{T_{k+1}}{\sum_{\tau=1}^{k+1} T_\tau}\left(-LN\left(1-\psi_{\sigma_{k+1}}^{2U}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}}, \sqrt{1 - exp^{-\left(\frac{T_{k+1}}{\sum_{\tau=1}^{k+1} T_\tau}\left(-LN\left(1-\psi_{\sigma_{k+1}}^{2U}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}} \right], \\ \left[\sqrt{exp^{-\left(\frac{T_{k+1}}{\sum_{\tau=1}^{k+1} T_\tau}\left(-LN\left(1-\mathcal{M}_{\sigma_{k+1}}^{2l}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}}, \sqrt{exp^{-\left(\frac{T_{k+1}}{\sum_{\tau=1}^{k+1} T_\tau}\left(-LN\left(1-\mathcal{M}_{\sigma_{k+1}}^{2U}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}} \right] \end{array} \right)$$

$$= \left(\begin{array}{c} \left[\sqrt{1 - exp^{-\left(\sum_{\tau=1}^{k+1} \frac{T_\tau}{\sum_{\tau=1}^{k+1} T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}}, \sqrt{1 - exp^{-\left(\sum_{\tau=1}^{k+1} \frac{T_\tau}{\sum_{\tau=1}^{k+1} T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}} \right], \\ \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{k+1} \frac{T_\tau}{\sum_{\tau=1}^{k+1} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{k+1} \frac{T_\tau}{\sum_{\tau=1}^{k+1} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}} \right] \end{array} \right).$$

Eq. (12) is therefore valid for $\emptyset = k + 1$.

We conclude that Eq. (12) holds for any value of $\emptyset$ because of the forms (I) and (II), proving that the IVPyFPAAA operator satisfies the requirements of boundedness, monotonicity, and idempotency. Any AO must have these characteristics in order to function correctly within expected bounds, behave predictably when input values change, and retain consistency when applied repeatedly to the same set of data.

Appendix-2: Proof of Theorem 2

Since $\sigma_\tau = \left(\left[\psi_{\sigma_\tau}^{l}, \psi_{\sigma_\tau}^{U}\right], \left[\mathcal{M}_{\sigma_\tau}^{l}, \mathcal{M}_{\sigma_\tau}^{U}\right]\right) = \sigma = \left(\left[\psi_{\sigma}^{l}, \psi_{\sigma}^{U}\right], \left[\mathcal{M}_{\sigma}^{l}, \mathcal{M}_{\sigma}^{l}\right]\right)$. Then, we have by Eq. (12) the following:

$$IVPyFPAAA(\sigma_1, \sigma_2, \ldots \sigma_\emptyset) = \left(\begin{array}{c} \left[\sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}}, \sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}} \right], \\ \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}} \right] \end{array} \right)$$

$$= \left(\left[\sqrt{1 - exp^{-\left(\left(-LN\left(1-\psi_{\sigma}^{2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}}, \sqrt{1 - exp^{-\left(\left(-LN\left(1-\psi_{\sigma}^{2U}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}} \right], \left[\sqrt{exp^{-\left(\left(-LN\left(\psi_{\sigma}^{2l}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}}, \sqrt{exp^{-\left(\left(-LN\left(\psi_{\sigma}^{2U}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}} \right] \right)$$

$$= \left(\left[\sqrt{1 - exp^{-LN\left(1-\psi_{\sigma}^{2l}\right)}}, \sqrt{1 - exp^{-LN\left(1-\psi_{\sigma}^{2U}\right)}} \right], \left[\sqrt{exp^{LN\psi_{\sigma}^{2l}}}, \sqrt{exp^{LN\psi_{\sigma}^{2U}}} \right] \right) = \left(\left[\sqrt{\psi_{\sigma}^{2l}}, \sqrt{\psi_{\sigma}^{2U}} \right], \left[\sqrt{\psi_{\sigma}^{2l}}, \sqrt{\psi_{\sigma}^{2U}} \right] \right) = \sigma.$$

Thus, Eq. (13) holds.

Appendix-3: Proof of Theorem 3

Let $\sigma_\tau = \left(\left[\psi^l_{\sigma_\tau}, \psi^U_{\sigma_\tau}\right], \left[\mathcal{M}^l_{\sigma_\tau}, \mathcal{M}^U_{\sigma_\tau}\right]\right)$ be a collection of PyFVs. Let $\sigma^- = min(\sigma_1, \beta_2, \ldots, \sigma_\emptyset) = (\psi^-_\sigma, \psi^-_\sigma)$ and $\sigma^+ = max(\sigma_1, \beta_2, \ldots, \sigma_\emptyset) = (\psi^+_\sigma, \psi^+_\sigma)$. Hence, there are the subsequent inequalities:

$$\left[\sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_\sigma^{-2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_\sigma^{-2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}\right]$$

$$\leq \left[\sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}\right]$$

$$\leq \left[\sqrt{1 - exp^{-\left(\sum_{\tau=1}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_\sigma^{+2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1 - exp^{-\left(\sum_{\tau=1}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_\sigma^{+2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}\right]$$

$$\left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_\sigma^{+2l}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_\sigma^{+2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}\right]$$

$$\geq \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\psi_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\psi_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}\right]$$

$$\geq \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_\sigma^{-2l}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_\sigma^{-2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}\right]$$

Therefore, $\sigma^- \leq IFPAAA(\sigma_1, \sigma_2, \ldots, \sigma_\emptyset \leq \sigma^+)$.

Appendix-4: Proof of Theorem 5

First, we compute $IVPyFPAAA(\sigma_1 \oplus \alpha, \sigma_2 \oplus \alpha, \ldots, \sigma_\emptyset \oplus \alpha)$ as:

$$IVPyFPAAA(\sigma_1, \sigma_2, \ldots \sigma_\emptyset) = \left(\begin{array}{c}\left[\sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}\right], \\ \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}\right]\end{array}\right)$$

$$\sigma_\tau \oplus \alpha = \left(\begin{array}{c}\left[\sqrt{1 - exp^{-\left(\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc} + \left(-LN\left(1-\psi_\alpha^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1 - exp^{-\left(\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc} + \left(-LN\left(1-\psi_\alpha^{2U}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}\right], \\ \left[\sqrt{exp^{-\left(\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc} + \left(-LN\left(\mathcal{M}_\alpha^{2l}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}, \sqrt{exp^{-\left(\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc} + \left(-LN\left(\mathcal{M}_\alpha^{2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}\right]\end{array}\right)$$

$$IVPyFPAAA(\sigma_1 \oplus \alpha, \sigma_2 \oplus \alpha, \dots, \sigma_\emptyset \oplus \alpha) = \left(\begin{array}{c} \left[\begin{array}{c} \sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(1 - \left(1 - exp^{-\left(\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}} + \left(-LN\left(1-\psi_{\alpha}^{2l}\right)\right)^{\mathbb{O}} \right)^{\frac{1}{\mathbb{O}}}} \right) \right) \right)^{\mathbb{O}} \right)^{\frac{1}{\mathbb{O}}}}}, \\ \sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(1 - \left(1 - exp^{-\left(\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}} + \left(-LN\left(1-\psi_{\alpha}^{2U}\right)\right)^{\mathbb{O}} \right)^{\frac{1}{\mathbb{O}}}} \right) \right) \right)^{\mathbb{O}} \right)^{\frac{1}{\mathbb{O}}}}} \end{array} \right], \\ \left[\begin{array}{c} \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(exp^{-\left(\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}} + \left(-LN\left(\mathcal{M}_{\alpha}^{2l}\right)\right)^{\mathbb{O}} \right)^{\frac{1}{\mathbb{O}}}} \right) \right)^{\mathbb{O}} \right)^{\frac{1}{\mathbb{O}}}}}, \\ \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(exp^{-\left(\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}} + \left(-LN\left(\mathcal{M}_{\alpha}^{2U}\right)\right)^{\mathbb{O}} \right)^{1/\mathbb{O}}} \right) \right)^{\mathbb{O}} \right)^{1/\mathbb{O}}}} \end{array} \right] \end{array} \right)$$

$$IVPyFPAAA(\sigma_1 \oplus \alpha, \sigma_2 \oplus \alpha, \dots, \sigma_\emptyset \oplus \alpha) =$$

$$\left(\begin{array}{c} \left[\sqrt{1 - exp^{-\left(\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}} \right) + \left(-LN\left(1-\left(\psi_{\alpha}^{2l}\right)\right)\right)^{\mathbb{O}} \right)^{\frac{1}{\mathbb{O}}}}}, \sqrt{1 - exp^{-\left(\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}} \right) + \left(-LN\left(1-\left(\psi_{\alpha}^{2U}\right)\right)\right)^{\mathbb{O}} \right)^{\frac{1}{\mathbb{O}}}}} \right], \\ \left[\sqrt{exp^{-\left(\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}} \right) + \left(-LN\left(\mathcal{M}_{\alpha}^{2l}\right)\right)^{\mathbb{O}} \right)^{1/\mathbb{O}}}}, \sqrt{exp^{-\left(\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}} \right) + \left(-LN\left(\mathcal{M}_{\alpha}^{2U}\right)\right)^{\mathbb{O}} \right)^{1/\mathbb{O}}}} \right] \end{array} \right)$$

$$IVPyFPAAA(\sigma_1 \oplus \alpha, \sigma_2 \oplus \alpha, \dots, \sigma_\emptyset \oplus \alpha) =$$

$$\left(\begin{array}{c} \left[\sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(1-\psi_{\sigma_\tau \oplus \alpha}^{2l}\right)\right)^{\mathbb{O}} \right)^{\frac{1}{\mathbb{O}}}}}, \sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(1-\psi_{\sigma_\tau \oplus \alpha}^{2U}\right)\right)^{\mathbb{O}} \right)^{\frac{1}{\mathbb{O}}}}} \right], \\ \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(\mathcal{M}_{\sigma_\tau \oplus \alpha}^{2l}\right)\right)^{\mathbb{O}} \right)^{1/\mathbb{O}}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(\mathcal{M}_{\sigma_\tau \oplus \alpha}^{2U}\right)\right)^{\mathbb{O}} \right)^{1/\mathbb{O}}}} \right] \end{array} \right)$$

Now we make an expression for $IVPyFPAAA(\sigma_1, \sigma_2, \dots, \sigma_\emptyset) \oplus \alpha$ as:

$$IVPyFPAAA(\sigma_1, \sigma_2, \dots \sigma_\emptyset) = \left(\begin{array}{c} \left[\sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}} \right)^{\frac{1}{\mathbb{O}}}}}, \sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}} \right)^{\frac{1}{\mathbb{O}}}}} \right], \\ \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}} \right)^{1/\mathbb{O}}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}} \right)^{1/\mathbb{O}}}} \right] \end{array} \right)$$

$$IVIFPAAA(\sigma_1,\sigma_2,\dots\sigma_\emptyset)\oplus\alpha=\left(\begin{array}{c}\left[\sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}},\sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}\right]\\ \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}},\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}\right]\end{array}\right)\oplus$$

$$\left(\left[\sqrt{\psi_\alpha^{2l}},\sqrt{\psi_\alpha^{2U}}\right],\left[\sqrt{\mathcal{M}_\alpha^{2l}},\sqrt{\mathcal{M}_\alpha^{2U}}\right]\right)$$

$$IVPyFPAAA(\sigma_1,\sigma_2,\dots\sigma_\emptyset)\oplus\alpha=\left(\begin{array}{c}\left[\begin{array}{c}\sqrt{1-exp^{-\left(\left(-LN\left(1-\left(1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}\right)\right)\right)^{\circledcirc}+\left(-LN\left(1-\left(\psi_\alpha^{2l}\right)\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}},\\ \sqrt{1-exp^{-\left(\left(-LN\left(1-\left(1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}\right)\right)\right)^{\circledcirc}+\left(-LN\left(1-\left(\psi_\alpha^{2U}\right)\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}\end{array}\right],\\ \left[\begin{array}{c}\sqrt{exp^{-\left(\left(-LN\left(exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}\right)\right)^{\circledcirc}+\left(-LN\left(\mathcal{M}_\alpha^{2l}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}},\\ \sqrt{exp^{-\left(\left(-LN\left(exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}\right)\right)^{\circledcirc}+\left(-LN\left(\mathcal{M}_\alpha^{2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}\end{array}\right]\end{array}\right)$$

$$=\left(\begin{array}{c}\left[\begin{array}{c}\sqrt{1-exp^{-\left(\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)+\left(-LN\left(1-\left(\psi_\alpha^{2l}\right)\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}},\\ \sqrt{1-exp^{-\left(\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)+\left(-LN\left(1-\left(\psi_\alpha^{2U}\right)\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}\end{array}\right],\\ \left[\begin{array}{c}\sqrt{exp^{-\left(\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)+\left(-LN\left(\mathcal{M}_\alpha^{2l}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}},\\ \sqrt{exp^{-\left(\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)+\left(-LN\left(\mathcal{M}_\alpha^{2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}\end{array}\right]\end{array}\right)$$

Hence, $IVPyFPAAA(\sigma_1\oplus\alpha,\sigma_2\oplus\alpha,\dots,\sigma_\emptyset\oplus\alpha)=IVPyFPAAA(\sigma_1,\sigma_2,\dots,\sigma_\emptyset)\oplus\alpha$.

Appendix-5: Proof of Theorem 6

We have the following based on Eq. (8):

$$\varsigma\sigma=\left(\begin{array}{c}\left[\sqrt{1-exp^{-\left(\varsigma\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}},\sqrt{1-exp^{-\left(\varsigma\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}\right],\\ \left[\sqrt{exp^{-\left(\varsigma\left(-LN\left(\psi_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}},\sqrt{exp^{-\left(\varsigma\left(-LN\left(\psi_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}\right]\end{array}\right)$$

According to Theorem 1, we have:

$$IVPyFPAAA(\varsigma\sigma_1,\varsigma\sigma_2,\dots,\varsigma\sigma_\emptyset)$$

$$=\left(\begin{array}{c}\left[\sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\left(1-exp^{-\left(\varsigma\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}\right)\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}},\sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\left(1-exp^{-\left(\varsigma\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}\right)\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}\right],\\ \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(exp^{-\left(\varsigma\left(-LN\left(\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}},\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(exp^{-\left(\varsigma\left(-LN\left(\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}}\right]\end{array}\right)$$

$$=\left(\begin{array}{c}\left[\sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(\left(\varsigma\left(-\ln\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{0}}\right)\right)\right)^{\frac{1}{\mathbb{0}}}}},\sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(\left(\varsigma\left(-\ln\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{0}}\right)\right)\right)^{\frac{1}{\mathbb{0}}}}}\right],\\ \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(\left(\varsigma\left(-\ln\left(\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{0}}\right)\right)\right)^{1/\mathbb{0}}}},\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(\left(\varsigma\left(-\ln\left(\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{0}}\right)\right)\right)^{1/\mathbb{0}}}}\right]\end{array}\right)$$

$$\varsigma\mathrm{IV}PyFPAAA(\sigma_1,\sigma_2,\dots,\sigma_\emptyset)=\varsigma\left(\begin{array}{c}\left[\sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}},\sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}\right],\\ \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}},\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}\right]\end{array}\right)$$

$$=\left(\begin{array}{c}\left[\sqrt{1-exp^{-\left(\varsigma\left(-LN\left(1-\left(1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}\right)\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}},\sqrt{1-exp^{-\left(\varsigma\left(-LN\left(1-\left(1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}\right)\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}\right],\\ \left[\sqrt{exp^{-\left(\varsigma\left(-LN\left(exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}},\sqrt{exp^{-\left(\varsigma\left(-LN\left(exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}}\right]\end{array}\right)$$

$$= \left(\begin{array}{c} \left[\sqrt{1 - exp^{-\left(\varsigma\left(\left(\sum_{j=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-\ln\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)\right)\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1 - exp^{-\left(\varsigma\left(\left(\sum_{j=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-\ln\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)\right)\right)^{\frac{1}{\circledcirc}}}} \right], \\ \left[\sqrt{exp^{-\left(\varsigma\left(\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-\ln\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)\right)\right)^{1/\circledcirc}}}, \sqrt{exp^{-\left(\varsigma\left(\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-\ln\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)\right)\right)^{1/\circledcirc}}} \right] \end{array} \right)$$

Hence, $IVPyFPAAA(\varsigma\sigma_1, \varsigma\sigma_2, \ldots, \varsigma\sigma_\emptyset) = \varsigma \mathrm{IV}PyFPAAA(\sigma_1, \sigma_2, \ldots, \sigma_\emptyset)$.

Appendix-6: Proof of Theorem 7

$IVPyFPAAA(\varsigma\sigma_1 \oplus \alpha, \varsigma\sigma_2 \oplus \alpha, \ldots, \varsigma\sigma_\emptyset \oplus \alpha) = \varsigma \mathrm{IV}PyFPAAA(\sigma_1, \sigma_2, \ldots, \sigma_\emptyset) \oplus \alpha$

$$IVPyFPAAA(\sigma_1, \sigma_2, \ldots \sigma_\emptyset) = \left(\begin{array}{c} \left[\sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}} \right], \\ \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}} \right] \end{array} \right)$$

$$\varsigma IVPyFPAAA(\sigma_1, \sigma_2, \ldots, \sigma_\emptyset) = \varsigma \left(\begin{array}{c} \left[\sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}} \right], \\ \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}} \right] \end{array} \right)$$

$$\varsigma \mathrm{IV}PyFPAAA(\sigma_1, \sigma_2, \ldots, \sigma_\emptyset) = \left(\begin{array}{c} \left[\sqrt{1 - exp^{-\left(\varsigma\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1 - exp^{-\left(\varsigma\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)\right)^{\frac{1}{\circledcirc}}}} \right], \\ \left[\sqrt{exp^{-\left(\varsigma\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)\right)^{1/\circledcirc}}}, \sqrt{exp^{-\left(\varsigma\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)\right)^{1/\circledcirc}}} \right] \end{array} \right)$$

$\varsigma \mathrm{IV}PyFPAAA(\sigma_1, \sigma_2, \ldots, \sigma_\emptyset) \oplus \alpha =$

$$\left(\begin{array}{c} \left[\sqrt{1 - exp^{-\left(\varsigma\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1 - exp^{-\left(\varsigma\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)\right)^{\frac{1}{\circledcirc}}}} \right], \\ \left[\sqrt{exp^{-\left(\varsigma\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)\right)^{1/\circledcirc}}}, \sqrt{exp^{-\left(\varsigma\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)\right)^{1/\circledcirc}}} \right] \end{array} \right) \oplus \left(\left[\sqrt{\psi_\alpha^{2l}}, \sqrt{\psi_\alpha^{2U}}\right], \left[\sqrt{\mathcal{M}_\alpha^{2l}}, \sqrt{\mathcal{M}_\alpha^{2U}}\right]\right)$$

$$\varsigma \mathrm{IV}PyFPAAA(\sigma_1, \sigma_2, \ldots, \sigma_{\emptyset}) \oplus \alpha =$$

$$\left(\begin{array}{c}\left[\sqrt{1 - exp^{-\left(\left(\varsigma\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{0}}\right)\right)+\left(-LN\left(1-\psi_{\alpha}^{2l}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}, \sqrt{1 - exp^{-\left(\left(\varsigma\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{0}}\right)\right)+\left(-LN\left(1-\psi_{\alpha}^{2U}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}\right], \\ \left[\sqrt{exp^{-\left(\left(\left(\varsigma\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{0}}\right)\right)\right)+\left(-LN\left(\psi_{\alpha}^{2l}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}}, \sqrt{exp^{-\left(\left(\left(\varsigma\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{0}}\right)\right)\right)+\left(-LN\left(\psi_{\alpha}^{2U}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}}\right]\end{array}\right)$$

$$IVPyFPAAA(\varsigma\sigma_1 \oplus \alpha, \varsigma\sigma_2 \oplus \alpha, \ldots, \varsigma\sigma_{\emptyset} \oplus \alpha)$$

$$\varsigma\sigma_\tau = \left(\begin{array}{c}\left[\sqrt{1 - exp^{-\left(\varsigma\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}, \sqrt{1 - exp^{-\left(\varsigma\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}\right], \\ \left[\sqrt{exp^{-\left(\varsigma\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}}, \sqrt{exp^{-\left(\varsigma\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}}\right]\end{array}\right)$$

$$\varsigma\sigma_\tau \oplus \alpha = \left(\begin{array}{c}\left[\sqrt{1 - exp^{-\left(\varsigma\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}, \sqrt{1 - exp^{-\left(\varsigma\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}\right], \\ \left[\sqrt{exp^{-\left(\varsigma\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}}, \sqrt{exp^{-\left(\varsigma\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}}\right]\end{array}\right) \oplus \left([\psi_\alpha^l, \psi_\alpha^U], [\psi_\alpha^l, \psi_\alpha^U]\right)$$

$$\varsigma\sigma_\tau \oplus \alpha = \left(\begin{array}{c}\left[\begin{array}{c}\sqrt{1 - exp^{-\left(\left(-LN\left(1-\left(1-exp^{-\left(\varsigma\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}\right)\right)\right)^{\mathbb{0}}+\left(-LN\left(1-\psi_{\alpha}^{2l}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}, \\ \sqrt{1 - exp^{-\left(\left(-LN\left(1-\left(1-exp^{-\left(\varsigma\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}\right)\right)\right)^{\mathbb{0}}+\left(-LN\left(1-\psi_{\alpha}^{2U}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}\end{array}\right], \\ \left[\begin{array}{c}\sqrt{exp^{-\left(\left(-LN\left(exp^{-\left(\varsigma\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}\right)\right)^{\mathbb{0}}+\left(-LN\left(\psi_{\alpha}^{2l}\right)\right)^{\mathbb{0}}\right)^{\frac{1}{\mathbb{0}}}}}, \\ \sqrt{exp^{-\left(\left(-LN\left(exp^{-\left(\varsigma\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}\right)\right)^{\mathbb{0}}+\left(-LN\left(\psi_{\alpha}^{2U}\right)\right)^{\mathbb{0}}\right)^{1/\mathbb{0}}}}\end{array}\right]\end{array}\right)$$

$$
= \left(\begin{array}{c}
\left[\sqrt{1-exp\left(-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\left(1-exp^{-\left(\left(-LN\left(1-\left(1-exp^{-\left(\varsigma\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}\right)\right)\right)^{\mathbb{O}}+\left(-LN\left(1-\psi_{\alpha}^{2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}\right)\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}\right)}, \right. \\
\left. \sqrt{1-exp\left(-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\left(1-exp^{-\left(\left(-LN\left(1-\left(1-exp^{-\left(\varsigma\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}\right)\right)\right)^{\mathbb{O}}+\left(-LN\left(1-\psi_{\alpha}^{2U}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}\right)\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}\right)}, \right] \\
\left[\sqrt{exp\left(-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(exp^{-\left(\left(-LN\left(exp^{-\left(\varsigma\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}\right)\right)^{\mathbb{O}}+\left(-LN\left(\psi_{\alpha}^{2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}\right)}, \right. \\
\left. \sqrt{exp\left(-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(exp^{-\left(\left(-LN\left(exp^{-\left(\varsigma\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}\right)\right)^{\mathbb{O}}+\left(-LN\left(\psi_{\alpha}^{2U}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}\right)} \right]
\end{array} \right)
$$

$$
\left(\begin{array}{c}
\left[\sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(\varsigma\left(\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}}\right)\right)+\left(-LN\left(1-\psi_{\alpha}^{2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}}, \sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(\varsigma\left(\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}}\right)\right)+\left(-LN\left(1-\psi_{\alpha}^{2U}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}} \right], \\
\left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(\left(\varsigma\left(\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}}\right)\right)\right)+\left(-LN\left(\psi_{\alpha}^{2l}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(\left(\varsigma\left(\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}}\right)\right)\right)+\left(-LN\left(\psi_{\alpha}^{2U}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}} \right]
\end{array} \right)
$$

Appendix-7: Proof of Theorem 8

According to Theorem 1, we have:

$$
\sigma_\tau \oplus \alpha_\tau = \left(\begin{array}{c}
\left[\sqrt{1-exp^{-\left(\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}}+\left(-LN\left(1-\psi_{\alpha_\tau}^{2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}}, \sqrt{1-exp^{-\left(\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}}+\left(-LN\left(1-\psi_{\alpha_\tau}^{2U}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}} \right], \\
\left[\sqrt{exp^{-\left(\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}}+\left(-LN\left(\mathcal{M}_{\alpha_\tau}^{2l}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}}, \sqrt{exp^{-\left(\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}}+\left(-LN\left(\mathcal{M}_{\alpha_\tau}^{2U}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}} \right]
\end{array} \right)
$$

$$IVPyFPAAA(\sigma_1,\sigma_2,\dots,\sigma_\emptyset) = \left(\begin{array}{c}\left[\sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}\right], \\ \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}\right]\end{array}\right)$$

$$IVPyFPAAA(\sigma_1 \oplus \alpha_1, \sigma_2 \oplus \alpha_2, \dots, \sigma_\emptyset \oplus \alpha_\emptyset) =$$
$$\left(\begin{array}{c}\left[\sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(\left(\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}+\left(-LN\left(1-\psi_{\alpha_\tau}^{2l}\right)\right)^{\circledcirc}\right)\right)\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(\left(\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}+\left(-LN\left(1-\psi_{\alpha_\tau}^{2U}\right)\right)^{\circledcirc}\right)\right)\right)^{\frac{1}{\circledcirc}}}}\right], \\ \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(\left(\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}+\left(-LN\left(\mathcal{M}_{\alpha_\tau}^{2l}\right)\right)^{\circledcirc}\right)\right)\right)^{1/\circledcirc}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(\left(\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}+\left(-LN\left(\mathcal{M}_{\alpha_\tau}^{2U}\right)\right)^{\circledcirc}\right)\right)\right)^{1/\circledcirc}}}\right]\end{array}\right)$$

$$IVPyFPAAA(\sigma_1,\sigma_2,\dots,\sigma_\emptyset) \oplus IVPyFPAAA(\alpha_1,\alpha_2,\dots,\alpha_\emptyset) =$$
$$\left(\begin{array}{c}\left(\begin{array}{c}\left[\sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}\right], \\ \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}\right]\end{array}\right) \oplus \\ \left(\begin{array}{c}\left[\sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\alpha_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\alpha_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}\right], \\ \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\alpha_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\alpha_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}\right]\end{array}\right)\end{array}\right)$$

$$
= \left(\begin{array}{c} \left[\sqrt{1 - exp^{-\left(\left(-LN\left(1 - \left(1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(1 - \psi_{\sigma_\tau}^{2l} \right) \right)^{\mathbb{D}} \right)^{\frac{1}{\mathbb{D}}}} \right) \right) \right)^{\mathbb{D}} + \left(-LN\left(1 - \left(1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(1 - \psi_{\alpha_\tau}^{2l} \right) \right)^{\mathbb{D}} \right)^{\frac{1}{\mathbb{D}}}} \right) \right) \right)^{\mathbb{D}} \right)^{\frac{1}{\mathbb{D}}}}}, \right. \\ \left. \sqrt{1 - exp^{-\left(\left(-LN\left(1 - \left(1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(1 - \psi_{\sigma_\tau}^{2U} \right) \right)^{\mathbb{D}} \right)^{\frac{1}{\mathbb{D}}}} \right) \right) \right)^{\mathbb{D}} + \left(-LN\left(1 - \left(1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(1 - \psi_{\alpha_\tau}^{2U} \right) \right)^{\mathbb{D}} \right)^{\frac{1}{\mathbb{D}}}} \right) \right) \right)^{\mathbb{D}} \right)^{\frac{1}{\mathbb{D}}}}} \right], \\ \left[\sqrt{exp^{-\left(\left(-LN\left(exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l} \right) \right)^{\mathbb{D}} \right)^{\frac{1}{\mathbb{D}}}} \right) \right)^{\mathbb{D}} + \left(-LN\left(exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(\mathcal{M}_{\alpha_\tau}^{2l} \right) \right)^{\mathbb{D}} \right)^{\frac{1}{\mathbb{D}}}} \right) \right)^{\mathbb{D}} \right)^{\frac{1}{\mathbb{D}}}}}, \right. \\ \left. \sqrt{exp^{-\left(\left(-LN\left(exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U} \right) \right)^{\mathbb{D}} \right)^{\frac{1}{\mathbb{D}}}} \right) \right)^{\mathbb{D}} + \left(-LN\left(exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(\mathcal{M}_{\alpha_\tau}^{2U} \right) \right)^{\mathbb{D}} \right)^{1/\mathbb{D}}} \right) \right)^{\mathbb{D}} \right)^{1/\mathbb{D}}}} \right] \end{array} \right)
$$

$$
= \left(\begin{array}{c} \left[\sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(\left(-LN\left(1 - \psi_{\sigma_\tau}^{2l} \right) \right)^{\mathbb{D}} + \left(-LN\left(1 - \psi_{\alpha_\tau}^{2l} \right) \right)^{\mathbb{D}} \right) \right)^{\frac{1}{\mathbb{D}}}}}, \sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(\left(-LN\left(1 - \psi_{\sigma_\tau}^{2U} \right) \right)^{\mathbb{D}} + \left(-LN\left(1 - \psi_{\alpha_\tau}^{2U} \right) \right)^{\mathbb{D}} \right) \right)^{\frac{1}{\mathbb{D}}}}} \right], \\ \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(\left(\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l} \right) \right)^{\mathbb{D}} + \left(-LN\left(\mathcal{M}_{\alpha_\tau}^{2l} \right) \right)^{\mathbb{D}} \right) \right) \right)^{1/\mathbb{D}}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset} \frac{T_\tau}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(\left(\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U} \right) \right)^{\mathbb{D}} + \left(-LN\left(\mathcal{M}_{\alpha_\tau}^{2U} \right) \right)^{\mathbb{D}} \right) \right) \right)^{1/\mathbb{D}}}} \right] \end{array} \right)
$$

Appendix-8: Proof of Theorem 9

Theorem 9 can be demonstrated by using the mathematical induction approach.

For $\emptyset = 2$, using Aczel-Alsina operations of IVPyFVs, we obtain:

$$
\sigma_1^{\frac{T_1}{\sum_{\tau=1}^{\emptyset} T_\tau}} = \left(\begin{array}{c} \left[\sqrt{exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(\mathcal{M}_{\sigma_1}^{2l} \right) \right)^{\mathbb{D}} \right)^{\frac{1}{\mathbb{D}}}}}, \sqrt{exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(\mathcal{M}_{\sigma_1}^{2U} \right) \right)^{\mathbb{D}} \right)^{\frac{1}{\mathbb{D}}}}} \right], \\ \left[\sqrt{1 - exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(1 - \psi_{\sigma_1}^{2l} \right) \right)^{\mathbb{D}} \right)^{\frac{1}{\mathbb{D}}}}}, \sqrt{1 - exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(1 - \psi_{\sigma_1}^{2U} \right) \right)^{\mathbb{D}} \right)^{1/\mathbb{D}}}} \right] \end{array} \right)
$$

$$
\sigma_2^{\frac{T_2}{\sum_{\tau=1}^{\emptyset} T_\tau}} = \left(\begin{array}{c} \left[\sqrt{exp^{-\left(\frac{T_2}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(\mathcal{M}_{\sigma_2}^{2l} \right) \right)^{\mathbb{D}} \right)^{\frac{1}{\mathbb{D}}}}}, \sqrt{exp^{-\left(\frac{T_2}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(\mathcal{M}_{\sigma_2}^{2U} \right) \right)^{\mathbb{D}} \right)^{\frac{1}{\mathbb{D}}}}} \right], \\ \left[\sqrt{1 - exp^{-\left(\frac{T_2}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(1 - \psi_{\sigma_2}^{2l} \right) \right)^{\mathbb{D}} \right)^{\frac{1}{\mathbb{D}}}}}, \sqrt{1 - exp^{-\left(\frac{T_2}{\sum_{\tau=1}^{\emptyset} T_\tau} \left(-LN\left(1 - \psi_{\sigma_2}^{2U} \right) \right)^{\mathbb{D}} \right)^{1/\mathbb{D}}}} \right] \end{array} \right)
$$

$$
IVPyFPAAG(\sigma_1, \sigma_2) = \sigma_1^{\frac{T_1}{\sum_{\tau=1}^{\emptyset} T_\tau}} \otimes \sigma_2^{\frac{T_2}{\sum_{\tau=1}^{\emptyset} T_\tau}}
$$

$$
= \left(\begin{array}{c} \left[\sqrt{exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_1}^{2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}}, \sqrt{exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_1}^{2U}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}} \right], \\ \left[\sqrt{1 - exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(1-\psi_{\sigma_1}^{2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}}, \sqrt{1 - exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(1-\psi_{\sigma_1}^{2U}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}} \right] \end{array} \right)
$$

$$
\otimes \left(\begin{array}{c} \left[\sqrt{exp^{-\left(\frac{T_2}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_2}^{2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}}, \sqrt{exp^{-\left(\frac{T_2}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_2}^{2U}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}} \right], \\ \left[\sqrt{1 - exp^{-\left(\frac{T_2}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(1-\psi_{\sigma_2}^{2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}}, \sqrt{1 - exp^{-\left(\frac{T_2}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(1-\psi_{\sigma_2}^{2U}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}} \right] \end{array} \right)
$$

$$
= \left(\begin{array}{c} \left[\sqrt{exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_1}^{2l}\right)\right)^{\mathbb{O}} + \frac{T_2}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_2}^{2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}}, \sqrt{exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_1}^{2U}\right)\right)^{\mathbb{O}} + \frac{T_2}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_2}^{2U}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}} \right], \\ \left[\sqrt{1 - exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(1-\psi_{\sigma_1}^{2l}\right)\right)^{\mathbb{O}} + \frac{T_2}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(1-\psi_{\sigma_2}^{2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}}, \sqrt{1 - exp^{-\left(\frac{T_1}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}} + \frac{T_2}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}} \right] \end{array} \right)
$$

$$
= \left(\begin{array}{c} \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}} \right], \\ \left[\sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}}, \sqrt{1 - exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{2} T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}} \right] \end{array} \right)
$$

Hence, Eq. (19) is true for $\emptyset = 2$.

If Eq. (19) holds true for $\emptyset = k$, then we have:

$$
IVPyFPAAG(\sigma_1, \sigma_2, \dots \sigma_k) = \otimes_{\tau=1}^{k} \sigma_\tau^{\frac{T_\tau}{\sum_{\tau=1}^{k} T_\tau}} =
$$

$$
\left(\begin{array}{c} \left[\sqrt{exp^{-\left(\sum_{\tau=1}^{k}\frac{T_\tau}{\sum_{\tau=1}^{k} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{k}\frac{T_\tau}{\sum_{\tau=1}^{k} T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}} \right], \\ \left[\sqrt{1 - exp^{-\left(\sum_{\tau=1}^{k}\frac{T_\tau}{\sum_{\tau=1}^{k} T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}}, \sqrt{1 - exp^{-\left(\sum_{\tau=1}^{k}\frac{T_\tau}{\sum_{\tau=1}^{k} T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}} \right] \end{array} \right)
$$

Now for $\emptyset = k + 1$, we get:

$$
IVPyFPAAG(\sigma_1, \sigma_2, \dots \sigma_{k+1}) = \otimes_{\tau=1}^{k} \sigma_\tau^{\frac{T_\tau}{\sum_{\tau=1}^{k+1} T_\tau}} \otimes \sigma_{\mathrm{k+1}}^{\frac{T_{k+1}}{\sum_{\tau=1}^{k+1} T_\tau}}
$$

$$\left(\begin{array}{c}\left[\sqrt{exp^{-\left(\sum_{\tau=1}^{k}\frac{T_\tau}{\sum_{\tau=1}^{k+1}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{k}\frac{T_\tau}{\sum_{\tau=1}^{k+1}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}\right], \\ \left[\sqrt{1-exp^{-\left(\sum_{\tau=1}^{k}\frac{T_\tau}{\sum_{\tau=1}^{k+1}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1-exp^{-\left(\sum_{\tau=1}^{k}\frac{T_\tau}{\sum_{\tau=1}^{k+1}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}\right]\end{array}\right)$$

$$\otimes\left(\begin{array}{c}\left[\sqrt{exp^{-\left(\frac{T_{k+1}}{\sum_{\tau=1}^{k+1}T_\tau}\left(-LN\left(1-\mathcal{M}_{\sigma_{k+1}}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{exp^{-\left(\frac{T_{k+1}}{\sum_{\tau=1}^{k+1}T_\tau}\left(-LN\left(1-\mathcal{M}_{\sigma_{k+1}}^{2U}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}\right], \\ \left[\sqrt{1-exp^{-\left(\frac{T_{k+1}}{\sum_{\tau=1}^{k+1}T_\tau}\left(-LN\left(1-\psi_{\sigma_{k+1}}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1-exp^{-\left(\frac{T_{k+1}}{\sum_{\tau=1}^{k+1}T_\tau}\left(-LN\left(1-\psi_{\sigma_{k+1}}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}\right]\end{array}\right)$$

$$\left(\begin{array}{c}\left[\sqrt{exp^{-\left(\sum_{\tau=1}^{k+1}\frac{T_\tau}{\sum_{\tau=1}^{k+1}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{k+1}\frac{T_\tau}{\sum_{\tau=1}^{k+1}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}\right], \\ \left[\sqrt{1-exp^{-\left(\sum_{\tau=1}^{k+1}\frac{T_\tau}{\sum_{\tau=1}^{k+1}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1-exp^{-\left(\sum_{\tau=1}^{k+1}\frac{T_\tau}{\sum_{\tau=1}^{k+1}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}\right]\end{array}\right)$$

Eq. (19) is therefore valid for $\emptyset = k+1$. Due to forms (I) and (II), we conclude that Eq. (19) holds for any value of $\emptyset$.

Appendix-9: Proof of Theorem 10

Since $\sigma_\tau = \left(\left[\psi_{\sigma_\tau}^{l}, \psi_{\sigma_\tau}^{U}\right], \left[\mathcal{M}_{\sigma_\tau}^{l}, \mathcal{M}_{\sigma_\tau}^{U}\right]\right) = \sigma = \left(\left[\psi_{\sigma}^{l}, \psi_{\sigma}^{U}\right], \left[, \mathcal{M}_{\alpha_\tau}^{l}, \mathcal{M}_{\alpha_\tau}^{U}\right]\right)$, then we have:

$$IVPyFPAAG(\sigma_1, \sigma_2, \ldots \sigma_\emptyset) = \left(\begin{array}{c}\left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}\right], \\ \left[\sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}\right]\end{array}\right)$$

$$= \left(\left[\sqrt{exp^{-\left(\left(-LN\left(\mathcal{M}_{\sigma}^{2l}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}, \sqrt{exp^{-\left(\left(-LN\left(\mathcal{M}_{\sigma}^{2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}\right], \left[\sqrt{1-exp^{-\left(\left(-LN\left(1-\psi_{\sigma}^{2l}\right)\right)^{\circledcirc}\right)^{\frac{1}{\circledcirc}}}}, \sqrt{1-exp^{-\left(\left(-LN\left(1-\psi_{\sigma}^{2U}\right)\right)^{\circledcirc}\right)^{1/\circledcirc}}}\right]\right)$$

$$= \left(\left[\sqrt{1-exp^{-LN\left(1-\psi_{\sigma}^{2l}\right)}}, \sqrt{1-exp^{-LN\left(1-\psi_{\sigma}^{2U}\right)}}\right], \left[exp^{LN\mathcal{M}_{\sigma}^{2l}}, exp^{LN\mathcal{M}_{\sigma}^{2U}}\right]\right) = \left(\left[\sqrt{\psi_{\sigma}^{2l}}, \sqrt{\psi_{\sigma}^{2U}}\right], \left[\sqrt{\mathcal{M}_{\sigma}^{2l}}, \sqrt{\mathcal{M}_{\sigma}^{2U}}\right]\right) = \sigma.$$

Thus, $IVPyFPAAG(\sigma_1, \sigma_2, \ldots, \sigma_\varrho) = \sigma$ holds.

Appendix-10: Proof of Theorem 11

Let $\sigma_\tau = \left(\left[\psi_{\sigma_\tau}^{l}, \psi_{\sigma_\tau}^{U}\right], \left[\mathcal{M}_{\sigma_\tau}^{l}, \mathcal{M}_{\sigma_\tau}^{U}\right]\right)$ be a collection of IVPyFVs. Let $\sigma^- = min(\sigma_1, \beta_2, \ldots, \sigma_\emptyset) = \left(\left[\psi_{\sigma}^{-l}, \psi_{\sigma}^{-l}\right], \left[\mathcal{M}_{\sigma}^{-U}, \mathcal{M}_{\sigma}^{-U}\right]\right)$ and $\sigma^+ = max(\sigma_1, \beta_2, \ldots, \sigma_\emptyset) = \left(\left[\psi_{\sigma}^{+l}, \psi_{\sigma}^{+l}\right], \left[\mathcal{M}_{\sigma}^{+l}, \mathcal{M}_{\sigma}^{+U}\right]\right)$. Hence, the subsequent inequalities are:

$$\left[\sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\mathcal{M}_{\sigma}^{-2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}},\sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\mathcal{M}_{\sigma}^{-2U}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}}\right]$$

$$\geq\left[\sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}},\sqrt{1-exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\psi_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}}\right]$$

$$\geq\left[\sqrt{1-exp^{-\left(\sum_{\tau=1}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\mathcal{M}_{\sigma}^{+2l}\right)\right)^{\mathbb{O}}\right)^{\frac{1}{\mathbb{O}}}}},\sqrt{1-exp^{-\left(\sum_{\tau=1}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(1-\mathcal{M}_{\sigma}^{+2U}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}}\right]$$

$$\left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\psi_{\sigma}^{+2l}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}},\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\psi_{\sigma}^{+2U}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}}\right]$$

$$\leq\left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2l}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}},\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\mathcal{M}_{\sigma_\tau}^{2U}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}}\right]$$

$$\leq\left[\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\psi_{\sigma}^{-2l}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}},\sqrt{exp^{-\left(\sum_{\tau=1}^{\emptyset}\frac{T_\tau}{\sum_{\tau=1}^{\emptyset}T_\tau}\left(-LN\left(\psi_{\sigma}^{-2U}\right)\right)^{\mathbb{O}}\right)^{1/\mathbb{O}}}}\right]$$

Therefore, $\sigma^{-} \leq IVIFPAAG(\sigma_1, \sigma_2, \ldots, \sigma_{\emptyset} \leq \sigma^{+})$.

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Conflicts of Interest

The author declares no conflicts of interest.

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