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Thermal Performance Analysis of Trihybrid Nanofluid Model for Advanced Thermal Management Applications

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ABSTRACT

Trihybrid nanofluids, which is a combination of three different types of nanoparticles dispersed in a base fluid, let to the recent advances among the advanced thermal transport media because of their excellent heat transfer behaviour as well as an improvement in the thermophysical properties. These fluids are very promising for engineering applications where effective thermal management is needed, such as in microelectronics cooling, renewable energy systems and biomedical devices. In the present work, an analysis is performed with respect to the unsteady flow and heat transfer of trihybrid nanofluid including the mixture of Al_2O_3 - TiO_2 - ZnO nanoparticles suspended in distilled water. A fractal-fractional model is developed to describe the viscous flow in complex time and space. The Laplace transform method is applied to give the exact solution of the model subjected to suitable initial and boundary conditions. The effects of the key parameters, namely the volume fraction, fractional and fractal order, thermal radiation and Schmidt number on the velocity, temperature and concentration profile are investigated in detail. The additions of three nanoparticles together greatly improves the thermal behaviour of the fluid. These results offer useful information for thermal system optimization in a range of technical and industrial applications where accurate heat control is essential.

1. Introduction

Heat transport in physical systems can be enhanced in many ways such as by changing flow regimes, modifying boundary conditions, or changing the thermophysical characteristics of the fluid, particularly its thermal conductivity. Researchers have proposed various methods to augment the heat transport performance of the fluids. Among these techniques, the suspension of solid particles is well acknowledged [1,2]. This idea was experimentally investigated at Argonne National Laboratory, USA. This concept quickly attracted the attention of the researchers. The homogeneous

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mixture of nanosized particles and base fluid is referred to as a nanofluid [3]. A comprehensive discussion on nanofluid's preparation and applications can be found in [4,5]. Murtaza *et al.* [6] investigated the influence of silver nanoparticles using Mobil oil as a base fluid. The authors employed the rotating frame of reference and Jeffery fluid model in their analysis. The mathematical model was solved using integral transform techniques. From their analysis, they reported that the thermal performance of the base fluid was enhanced by 14.59% after suspending silver nanoparticles at a volume fraction of 0.04. In another study, Murtaza *et al.* [7] investigated the thermal performance of engine oil in response to Cadmium telluride nanoparticles. They observed that the performance rate increased by 15.27% after suspending nanoparticles at a concentration of 4%. Although nanofluids have shown very productive performance in the augmentation of heat transfer rate, they also possess some limitations that prevent them from getting the desired outcomes in certain physical phenomena. Therefore, the development of nanofluid technology has not ended here; binary and ternary hybrid nanofluids are now being used to overcome the issues of unary (single nanoparticle) nanofluids. The study of nanofluids has advanced to include hybrid nanofluids, which are suspensions of multiple types of nanoparticles in low concentrations [8]. To connect the synergistic effect of nanoparticles and overcome the limitations of single nanoparticle suspensions, hybrid nanofluids were developed. The hybrid nanofluid improves thermal conductivity and heat transmission, making it suitable for cost-effective industrial and technical applications. Hybrid nanofluid offers promising and greater thermophysical characteristics compared to unary nanofluid. Therefore, hybrid nanofluids are considered more environmentally friendly and less energy intensive. Due to their higher thermal properties, they can be used in many industries and technological sectors to improve energy efficiency [9]. Waqas *et al.* [10] investigated the heat transfer performance in 2D hybrid nanofluid flow on a stretching sheet. Alghamdi *et al.* [11] evaluated the laminar, MHD hybrid nanofluid flow for the applications of blood artery medication. Keeping in view the effects of thermal radiation, viscous dissipation, and Joule heating, Murtaza *et al.* [12] examined the hybrid nanofluid flow in a confined geometry.

In fluid dynamics systems, classical derivatives are used to analyze velocity, pressure, and temperature gradient, and have been used in a wide range of problems. Despite being widely used, classical derivative models fall short when it comes to studying the viscoelastic properties, memory effects, heredity properties of the fluid, and anomalous diffusion in heat transfer. Classical models effectively studied the conduction of heat and fluid flow but lacked the flexibility for time-dependent behavior in nanofluid flow and multi-phase flow. Fractional models are very useful for modeling those phenomena that contain anomalous diffusion in heat transfer, time-dependent viscosity, viscoelastic behavior, and nanofluid flow [13,14]. FDs extend the concept of classical derivative by extending the order of the derivative of the function to non-integer order. FD allows the researchers to capture accurately those properties of the fluid that cannot be examined or predicted by the classical model. This concept has effectively studied the heat transfer mechanism compared to the classical model. Asjad *et al.* [15] calculated the solution to the heat transfer nanofluid flow model. The authors have formulated the mathematical model of nanofluid flow in terms of classical-order PDEs. For the generalization of the model, the authors have employed the time-fractional derivative operator and then found its solution with the Laplace transform technique. From their analysis, the authors have concluded that water-based nanofluid has a greater velocity profile compared to the other studied nanofluid. Ahmad *et al.* [16] used the constant Caputo fractional derivative for the investigation of heat and mass transfer in viscous fluid flow. This analysis has been done for the response of uniform dispersion of single-wall carbon nanotubes in kerosene oil. From their analysis, they have observed that velocity, temperature, and mass diffusion are the increasing functions of fractional order α . This implies that increasing the value of fractional order enhances the velocity, temperature, and mass

diffusion fields. Some other interesting and significant studies can be found in [17-19]. Traditional calculus struggles to accurately model systems with irregular boundaries, such as porous materials, or spaces with abrupt changes. Similarly, classical fluid mechanics falls short when it comes to capturing the role of molecular size in fluid turbulence. To study fluid behavior on a molecular scale, it is essential to develop more suitable mathematical models. A relatively new approach introduces differentiation that incorporates both fractional-order derivatives and fractal dimensions. This concept is still evolving, and research in this area remains limited. Atangana [20] explored the connection between fractal and fractional calculus, highlighting how fractal-fractional differential and integral operators offer unique mathematical properties and practical numerical approximations. These models have been applied in various fields, including fluid flow through porous media, anomalous heat diffusion, and hybrid nanofluid behavior, offering a fresh perspective on complex physical processes [21,22]. These models consider the fractal nature of flow paths and the memory effects inherent in fractional calculus. The integration of fractal-fractional calculus in nanofluid models has provided more comprehensive insights into the effects of nanoparticle concentration, temperature distribution, and fluid-structure, improving the understanding of the enhancement of heat transfer in advanced cooling systems. Bearing in mind the applications of FFDs in nanofluid models, Khan *et al.* [23] used a fractal-fractional model for the examination of drilling nanofluid with clay nanoparticles. Zhen *et al.* [24] numerically simulated the fractal-fractional model for the analysis of silver and titanium oxide-based hybrid nanofluid in a channel. Abbas *et al.* [25] also utilized the fractal-fractional approach to investigate the heat and mass transfer in hybrid nanofluid flow. From all these analyses, it has been found that the fractal-fractional approach provides greater profiles for the analysis of fluid rheology, heat distribution, and diffusion of mass in the fluid.

To the best of our knowledge, based on review articles already published in the scientific community on the fractal-fractional model, no one has examined the current new approach for nanofluid flow down a channel. Therefore, the authors took into consideration partially linked Newtonian nanofluid flow in a channel to address this shortcoming in the literature. The suggested problem's classical mathematical model was created using partial differential equations, which an operator for fractal-fractional derivative has generalized to take differential expressions into account. Additionally, the Laplace transform technique-based solution to the generalized model is given.

2. Methodology

2.1 Mathematical Model

This section outlines the formulation of underlying physical phenomena. The authors have considered the ternary hybrid nanofluid flow between two parallel vertical plates separated by distance d . The phenomenon is modeled considering the rheological properties of the Newtonian fluid. The flow is assumed to be incompressible and one-dimensional along the x -axis in the cartesian coordinate system. A uniform magnetic field of magnitude B_0 acts on a fluid flow in a transverse direction. The velocity field in vector notation is $\vec{V} = \{u(y, t), 0, 0\}$ considering all the assumptions.

According to the assumptions, the mathematical governing equations for trihybrid nanofluid for Newtonian fluid are given as:

$$\rho_{Thnf} \frac{\partial v(y, t)}{\partial t} = \mu_{Thnf} \frac{\partial^2 v(y, t)}{\partial y^2} - \left(\sigma_{Thnf} B_0^2 + \mu_{Thnf} \frac{\psi}{k_0} \right) v(y, t) + g(\rho\beta_T)_{Thnf} (\Theta(y, t) - T_\infty) + g(\rho\beta_c)_{Thnf} (\phi(y, t) - C_\infty), \quad (1)$$

$$(\rho C_p)_{Thnf} \frac{\partial \Theta(y,t)}{\partial t} = k_{Thnf} \frac{\partial^2 \Theta(y,t)}{\partial y^2} - \frac{\partial q_r}{\partial y}, \quad (2)$$

here $q_r = -\frac{4\omega_0}{3\beta_0} \frac{\partial T^4}{\partial y}$ and:

$$T^4 \approx (4T_\infty^3) \Theta(y,t) - 3T_\infty^4, \quad (3)$$

$$\frac{\partial \Phi(y,t)}{\partial t} = D_{Thnf} \frac{\partial^2 \Phi(y,t)}{\partial y^2}, \quad (4)$$

with:

$$\left. \begin{aligned} v(y,0) &= 0, & \Theta(y,0) &= T_\infty, & \Phi(y,0) &= C_\infty, \\ v(0,t) &= 0, & \Theta(0,t) &= T_\infty, & \Phi(0,t) &= C_\infty, \\ v(d,t) &= \sin \omega t, & \Theta(d,t) &= T_\infty, & \Phi(d,t) &= C_w \end{aligned} \right\}. \quad (5)$$

Table 1 represents the correlations between the base fluid (Distilled water) and the dispersed three different nanoparticles.

Table 1
Ternary nanofluid correlations

Properties	Correlations
Density	$\rho_{Thnf} = (1 - \phi_1) \left[(1 - \phi_2) \{ (1 - \phi_3) \rho_f + \phi_3 \rho_3 \} + \phi_2 \rho_2 \right] + \phi_1 \rho_1$
Viscosity	$\mu_{Thnf} = \frac{\mu_f}{(1 - \phi_3)^{2.5} (1 - \phi_2)^{2.5} (1 - \phi_1)^{2.5}}$
Volumetric expansion	$(\rho \beta_T)_{Thnf} = (1 - \phi_1) \left[(1 - \phi_2) \left\{ \frac{(1 - \phi_3)(\rho \beta_T)_f}{+ \phi_3 (\rho \beta_T)_3} \right\} + \phi_2 (\rho \beta_T)_2 \right] + \phi_1 (\rho \beta_T)_1$
Specific heat capacity	$(\rho C_p)_{Thnf} = (1 - \phi_1) \left[(1 - \phi_2) \left\{ \frac{(1 - \phi_3)(\rho C_p)_f}{+ \phi_3 (\rho C_p)_3} \right\} + \phi_2 (\rho C_p)_2 \right] + \phi_1 (\rho C_p)_1$
Thermal Conductivity	$\frac{k_{Thnf}}{k_{hnf}} = \frac{k_3 + 2k_{hnf} - 2\phi_3(k_{hnf} - k_3)}{k_3 + 2k_{hnf} + 2\phi_3(k_{hnf} - k_3)}, \frac{k_{hnf}}{k_{nf}} = \frac{k_2 + 2k_{nf} - 2\phi_2(k_{nf} - k_2)}{k_2 + 2k_{nf} + 2\phi_2(k_{nf} - k_2)}, \frac{k_{nf}}{k_f} = \frac{k_1 + 2k_f - 2\phi_1(k_f - k_1)}{k_1 + 2k_f + 2\phi_1(k_f - k_1)}$
Electrical conductivity	$\frac{\sigma_{Thnf}}{\sigma_{hnf}} = \frac{\sigma_3(1 + 2\phi_3) - \phi_{hnf}(1 - 2\phi_3)}{\sigma_3(1 + 2\phi_3) + \phi_{hnf}(1 - 2\phi_3)}, \frac{\sigma_{hnf}}{\sigma_{nf}} = \frac{\sigma_2(1 + 2\phi_2) - \phi_{nf}(1 - 2\phi_2)}{\sigma_2(1 + 2\phi_2) + \phi_{nf}(1 - 2\phi_2)}, \frac{\sigma_{nf}}{\sigma_f} = \frac{\sigma_1(1 + 2\phi_1) - \phi_f(1 - 2\phi_1)}{\sigma_1(1 + 2\phi_1) + \phi_f(1 - 2\phi_1)}$
Mass diffusivity	$D_{Thnf} = (1 - (\phi_3 + \phi_2 + \phi_1)) D_f$

Following are the dimensionless entities that make the system of equations dimensionless:

$$u = \frac{v}{U}, \quad \tau = \frac{v}{l^2} t, \quad \zeta = \frac{y}{l}, \quad T = \frac{\Theta - T_\infty}{T_w - T_\infty}, \quad C = \frac{\Phi - C_\infty}{C_w - C_\infty}. \quad (6)$$

Incorporate the trihybrid nanofluid correlations along with the dimensionless entities, Eqs. (1)-(4) will take the form:

$$\frac{\partial u(\zeta, \tau)}{\partial \tau} = \lambda_0 \frac{\partial^2 u(\zeta, \tau)}{\partial \zeta^2} - \left(M_0 + \frac{1}{K_0} \right) u(\zeta, \tau) + Gr_0 T(\zeta, \tau) + Gm_0 T(\zeta, \tau), \quad (7)$$

$$\frac{\partial T(\zeta, \tau)}{\partial \tau} = \Lambda_\alpha \frac{\partial^2 T(\zeta, \tau)}{\partial \zeta^2}, \quad (8)$$

$$\frac{\partial C(\zeta, \tau)}{\partial \tau} = \Lambda_\delta \frac{\partial^2 C(\zeta, \tau)}{\partial \zeta^2}, \quad (9)$$

and:

$$\left. \begin{aligned} u(\zeta, 0) &= 0, & T(\zeta, 0) &= 0, & C(\zeta, 0) &= 0, \\ u(0, \tau) &= 0, & T(0, \tau) &= 0, & C(0, \tau) &= 0, \\ u(d, \tau) &= \sin \tau, & T(d, \tau) &= 0, & C(d, \tau) &= 1 \end{aligned} \right\}. \quad (10)$$

with:

$$\lambda_0 = \frac{m_0}{m_1}, M_1 = M \lambda_1, \lambda_1 = \frac{d}{m_1}, Gr_0 = Gr \lambda_2, \lambda_2 = \frac{n_0}{m_1}, Gm_0 = Gm \lambda_3, \lambda_3 = \frac{n_1}{m_1}, \Lambda_\alpha = \frac{b_4}{b_0 \text{Pr}} [1 + Rd], \Lambda_\delta = \frac{a_0}{Sc}. \quad (11)$$

Constants depend on nanofluid correlations:

$$b_0 = (1 - \phi_1) \left[(1 - \phi_2) \left\{ (1 - \phi_3)_f + \phi_3 \frac{(\rho C_p)_3}{(\rho C_p)_f} \right\} + \phi_2 \frac{(\rho C_p)_2}{(\rho C_p)_f} \right] + \phi_1 \frac{(\rho C_p)_1}{(\rho C_p)_f}, \quad (12)$$

$$b_1 = \frac{k_3 + 2k_{hnf} - 2\phi_3(k_{hnf} - k_3)}{k_3 + 2k_{hnf} + 2\phi_3(k_{hnf} - k_3)}, \quad (13)$$

$$b_2 = \frac{k_2 + 2k_{nf} - 2\phi_2(k_{nf} - k_2)}{k_2 + 2k_{nf} + 2\phi_2(k_{nf} - k_2)}, \quad (14)$$

$$b_3 = \frac{k_1 + 2k_f - 2\phi_1(k_f - k_1)}{k_1 + 2k_f + 2\phi_1(k_f - k_1)}, \quad (15)$$

$$b_4 = b_1 b_2 b_3, \quad (16)$$

$$c_0 = 1 - (\phi_3 + \phi_2 + \phi_1), \quad (17)$$

$$d_1 = \frac{\sigma_3(1+2\phi_3) - \phi_{hf}(1-2\phi_3)}{\sigma_3(1+2\phi_3) + \phi_{hf}(1-2\phi_3)}, \quad (18)$$

$$d_2 = \frac{\sigma_2(1+2\phi_2) - \phi_{nf}(1-2\phi_2)}{\sigma_2(1+2\phi_2) + \phi_{nf}(1-2\phi_2)}, \quad (19)$$

$$d_3 = \frac{\sigma_1(1+2\phi_1) - \phi_f(1-2\phi_1)}{\sigma_1(1+2\phi_1) + \phi_f(1-2\phi_1)}, \quad (20)$$

$$d_4 = d_1 d_2 d_3, \quad (21)$$

$$m_0 = \frac{1}{(1-\phi_3)^{2.5} (1-\phi_2)^{2.5} (1-\phi_1)^{2.5}}, \quad (22)$$

$$m_1 = (1-\phi_1) \left[(1-\phi_2) \left\{ (1-\phi_3) + \phi_3 \frac{\rho_3}{\rho_f} \right\} + \phi_2 \frac{\rho_2}{\rho_f} \right] + \phi_1 \frac{\rho_1}{\rho_f}, \quad (23)$$

$$n_0 = \left[(1-\phi_1) \left[(1-\phi_2) \left\{ (1-\phi_3) + \phi_3 \frac{(\rho\beta_T)_3}{(\rho\beta_T)_f} \right\} + \phi_2 \frac{(\rho\beta_T)_2}{(\rho\beta_T)_f} \right] + \phi_1 \frac{(\rho\beta_T)_1}{(\rho\beta_T)_f} \right], \quad (24)$$

$$n_1 = \left[(1-\phi_1) \left[(1-\phi_2) \left\{ (1-\phi_3) + \phi_3 \frac{(\rho\beta_c)_3}{(\rho\beta_c)_f} \right\} + \phi_2 \frac{(\rho\beta_c)_2}{(\rho\beta_c)_f} \right] + \phi_1 \frac{(\rho\beta_c)_1}{(\rho\beta_c)_f} \right]. \quad (25)$$

In Table 2, we summarized all the dimensionless physical parameters that are obtained during the calculi of dimensional analysis.

Table 2

Nomenclature

Parameter	Mathematical expression	Parameter	Mathematical expression
Magnetic parameter	$M = \frac{\sigma_f B_0^2 \ell^2}{\rho_f U}$	Mass Grashoff number	$Gm = \frac{g(\beta_c)_f \ell^2 (C_w - C_\infty)}{U v_f}$
Porosity parameter	$\frac{1}{K} = \frac{v \psi \ell^2 \lambda_0}{k_0 U}$	Prandtl number	$Pr = \frac{\mu_f C_p}{k_f}$
Radiation parameter	$R_d = \frac{16 T_\infty^3 \omega_0}{3 \beta_0 k_f b_4}$	Thermal Grashoff number	$Gr = \frac{g(\beta_T)_f \ell^2 (T_w - T_\infty)}{U v_f}$
Schmidth number	$Sc = \frac{v_f}{D_f}$	Differential operator	$\delta(t)$

2.2 Generalization of the Classical Model

After replacing the time fractional operator of the fractal-fractional derivative, Eqs. (7)-(9) are transformed to:

$${}^{FF}D_t^{\alpha,\beta}u(\zeta,\tau) + \frac{u(\zeta,0)}{\Gamma(1-\alpha)}\tau^{-\alpha} = \beta\tau^{\beta-1} \left\{ \lambda_0 \frac{\partial^2 u(\zeta,\tau)}{\partial \zeta^2} - \left(M_0 + \frac{1}{K_0} \right) u(\zeta,\tau) \right. \\ \left. + Gr_0 T(\zeta,\tau) + Gm_0 C(\zeta,\tau) \right\}, \quad (26)$$

$${}^{FF}D_t^{\alpha,\beta}T(\zeta,\tau) + \frac{T(\zeta,0)}{\Gamma(1-\alpha)}\tau^{-\alpha} = \beta\tau^{\beta-1} \left\{ \Lambda_\alpha \frac{\partial T^2(\zeta,\tau)}{\partial \zeta^2} \right\}, \quad (27)$$

$${}^{FF}D_t^{\alpha,\beta}C(\zeta,\tau) + \frac{C(\zeta,0)}{\Gamma(1-\alpha)}\tau^{-\alpha} = \beta\tau^{\beta-1} \left\{ \Lambda_\delta \frac{\partial^2 C(\zeta,\tau)}{\partial \zeta^2} \right\}, \quad (28)$$

where ${}^{FF}D_t^{\alpha,\beta}$ is the fractal-fractional operator with exponential kernel [20].

2.3 Solution of the Fractal Fractional Model

Using Laplace transform technique, we arrived at:

$$s^\alpha u(\zeta,s) = \beta \Gamma(\beta) s^{-\beta} \left(\lambda_0 \frac{\partial^2 u(\zeta,s)}{\partial \zeta^2} - \left(M + \frac{1}{K} \right) u(\zeta,s) \right. \\ \left. + Gr_0 T(\zeta,s) + Gm_0 C(\zeta,s) \right), \quad (29)$$

$$s^\alpha T(\zeta,s) = \beta \Gamma(\beta) s^{-\beta} \left(\Lambda_\alpha \frac{\partial^2 T(\zeta,s)}{\partial \zeta^2} \right), \quad (30)$$

$$s^\alpha C(\zeta,s) = \beta \Gamma(\beta) s^{-\beta} \left(\Lambda_\delta \frac{\partial^2 C(\zeta,s)}{\partial \zeta^2} \right), \quad (31)$$

with:

$$\left. \begin{aligned} u(\zeta,0) &= 0, & T(\zeta,0) &= 0, & C(\zeta,0) &= 0, \\ u(0,s) &= 0, & T(0,s) &= 0, & C(0,s) &= 0, \\ u(1,s) &= \frac{\omega^2}{s^2 + \omega^2}, & T(1,s) &= \frac{1}{s}, & C(1,s) &= \frac{1}{s} \end{aligned} \right\}. \quad (32)$$

Solving Eqs. (29)-(31), we arrived at:

$$u(\zeta,s) = \frac{\omega^2}{s^2 + \omega^2} \frac{\sinh(\gamma \sqrt{\mathbb{R}s^\alpha})}{\sinh(\sqrt{\mathbb{R}s^\alpha})} - \frac{1}{s^{\alpha+1}} \left[\frac{Gr_1 \sinh(\gamma \sqrt{\Lambda_\alpha^0 s^\alpha})}{\sinh(\sqrt{\Lambda_\alpha^0 s^\alpha})} + \frac{Gm_1 \sinh(\gamma \sqrt{\Lambda_\delta^0 s^\alpha})}{\sinh(\sqrt{\Lambda_\delta^0 s^\alpha})} \right], \quad (33)$$

$$T(\zeta,s) = \frac{1}{s} \cdot \frac{\sinh(\gamma \sqrt{\Lambda_\alpha^0 s^\alpha})}{\sinh(\sqrt{\Lambda_\alpha^0 s^\alpha})}, \quad (34)$$

$$C(\zeta, s) = \frac{1}{s} \cdot \frac{\sinh\left(\gamma \sqrt{\Lambda_\delta^0 s^\alpha}\right)}{\sinh\left(\sqrt{\Lambda_\delta^0 s^\alpha}\right)}, \quad (35)$$

here $\gamma = \alpha + \beta$, $\mathbb{R} = \frac{1}{\beta \Gamma(\beta)} + M + \frac{1}{K}$, $\mathbb{R}_1 = \Lambda_\alpha^0 - \mathbb{R}$, $\mathbb{R}_2 = \Lambda_\delta^0 - \mathbb{R}$, $Gr_1 = \frac{Gr_0}{\mathbb{R}_1}$ and $Gm_1 = \frac{Gm_0}{\mathbb{R}_2}$.

Eqs. (33)-(35) are very complex and difficult to solve by any integral transformation means. It is therefore possible to find the inversion form using Zakian's numerical algorithms [26]:

$$u(\zeta, t) = \frac{2}{t} \sum_{i=1}^5 \Re e \left\{ k, \bar{u} \left(\zeta, \frac{\alpha_i}{t} \right) \right\}. \quad (36)$$

3. Results

In this part of the study, a comprehensive graphical interpretation of the obtained numerical results of the proposed model is presented (Figures 1-12). The influence of the key physical dimensionless parameters on the velocity, temperature, and concentration profile is depicted and interpreted. This parametric study helps to understand the system's sensitivity to transition from classical to fractional and fractal-fractional systems. Each parameter is discussed separately to highlight its influence on system behavior.

Figure 1, Figure 6, and Figure 9 show the effects of fractional order on the velocity, temperature, and concentration curves, respectively. It is evident from those figures that every profile is an increasing function of fractional order. As the fractional order value increases, all three profiles rise as well. The memory effect of the fluid is analyzed using a fractional order derivative. By simply substituting the classical time derivative function for the time fractional derivative operator in the governing equation, the fractional derivative can be used to analyze the system's memory effect.

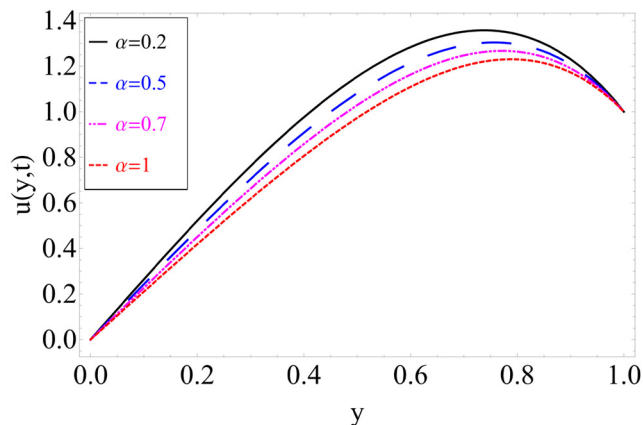


Fig. 1. Variation in velocity field against α

Figure 2, Figure 7, and Figure 10 show the effects of fractal order on the velocity, heat, and concentration profiles, respectively. The three illustrations make it obvious that all the profiles exhibit rising tendencies as the fractal dimension increases. The velocity profile of a fluid flow can be significantly impacted by fractal dimension. Notably, the existing model allows for the deduction of fractal, fractional, and integer order models by adjusting the parameters. Moreover, this model also provides multiple integral curves for the assessment of physical phenomena more deeply than the classical model.

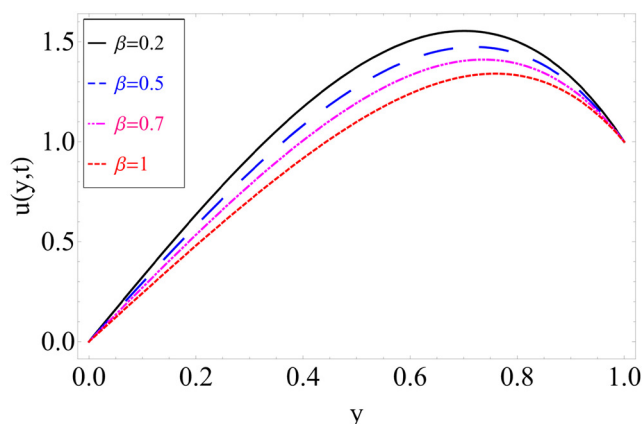


Fig. 2. Velocity profile against fractal dimension β

The thermal Grashof number G_r and mass Grashof number G_m plays a very significant role in natural convection flows, arising when the fluid's flow is affected by thermal and concentration gradients. G_r and G_m are a dimensionless physical entity that measures the ratio of buoyancy force to viscous force. Figure 3 and Figure 4 illustrate the influence of G_r and G_m on fluid flow behavior respectively.

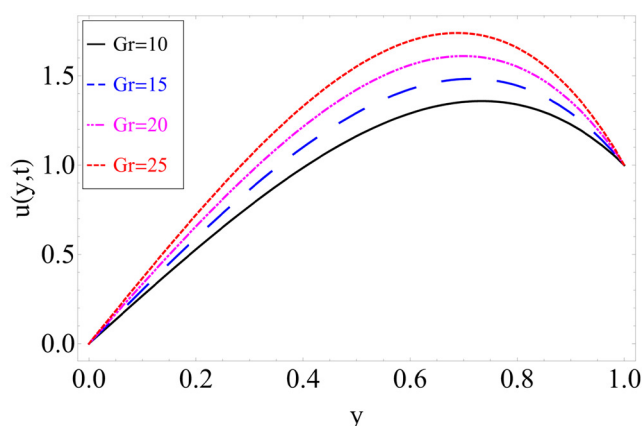


Fig. 3. Variation in velocity field against G_r

Figure 3 and Figure 4 show a rising trend with increasing values of G_r and G_m . Greater values of G_r indicate stronger natural convection effects, which consequently raise the temperature gradient and make the fluid less viscous, thereby accelerating fluid motion. Therefore, an increasing profile is observed in this figure. The parameter G_r is essential in all those phenomena where natural convection is used, such as the cooling of electronic devices, vertical heating plates, and the lubrication process. The same physical argument is also valid in the case of increasing values of G_m .

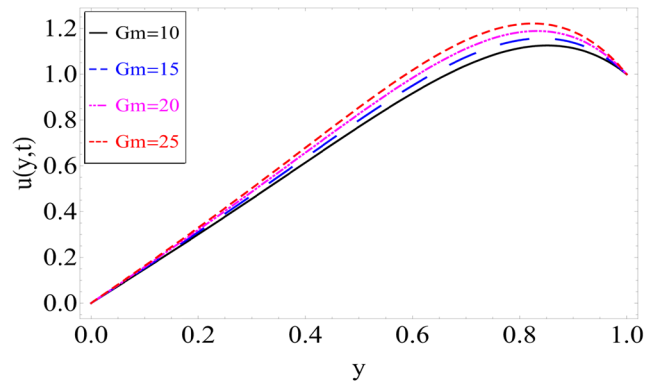


Fig. 4. Variation in velocity field against G_m

Figure 5 illustrates the impact of volume fraction ϕ of $Al_2O_3-TiO_2-ZnO$ on the velocity profile of a two-ternary hybrid nanofluid flow. It can be seen from the figure that the velocity profile of the hybrid nanofluid decreases with an increasing volume fraction ϕ .

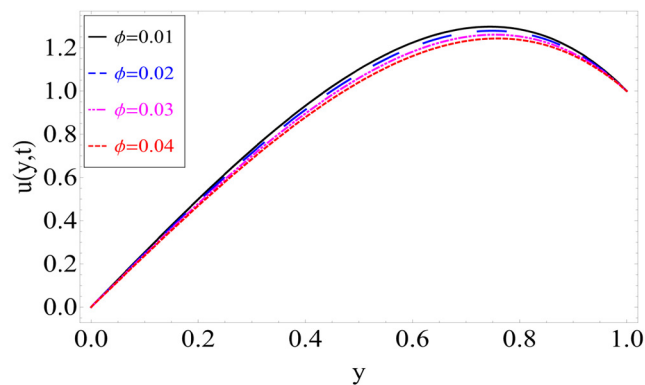


Fig. 5. Variation in velocity field against ϕ

Volume fraction indicates the ratio of the volume of the nanoparticles to the total volume of the fluid mixture. An increasing volume fraction leads to an increase in the effective viscosity and density of the fluid mixture. Consequently, the fluid becomes denser and more viscous which requires more energy to be set in motion. As a result, this slows down the entire flow and we see a decreasing trend in the velocity profile. Thus, the design of such nanofluid systems requires optimization of volume fraction to balance between heat transfer efficiency and flow performance (Figure 5).

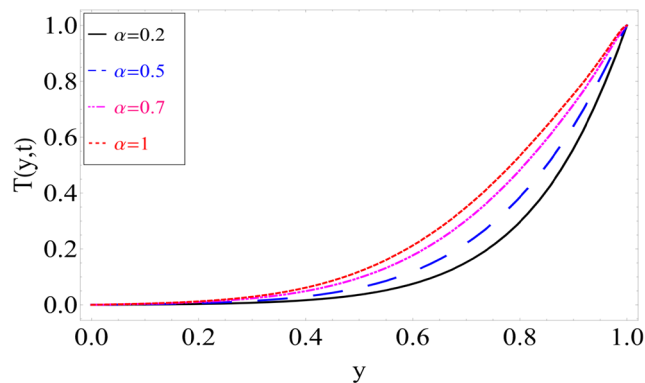


Fig. 6. Variation in temperature field against α

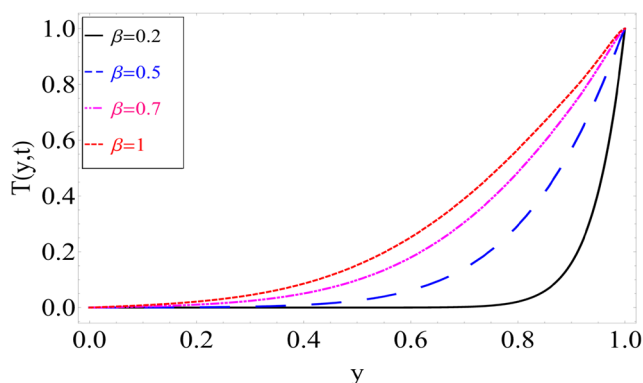


Fig. 7. Variation in temperature field against β

The volume fraction ϕ of $Al_2O_3-TiO_2-ZnO$ has a pronounced impact on the thermal profile of ternary hybrid nanofluid. These nanoparticles augment the heat transfer characteristics of the base fluid (distilled water) (Figure 8).

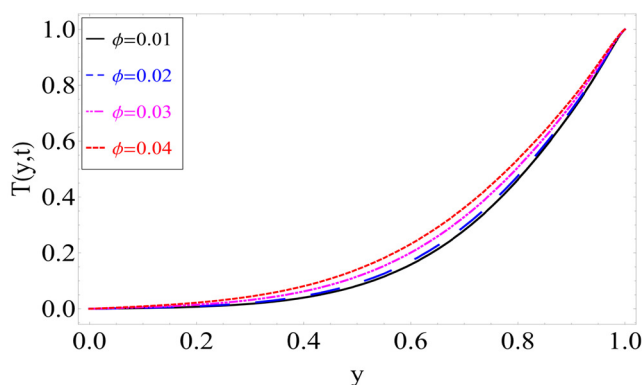


Fig. 8. Variation in temperature field against ϕ

All the nanoparticles $Al_2O_3-TiO_2-ZnO$ have higher thermal and physical properties, when added to the base fluid these particles create a faster thermal diffusion pathway allowing the heat to diffuse efficiently through the fluid. These nanoparticles play the role of a thermal bridge, allowing the thermal energy transfer via both the fluid molecules and solid particles. Therefore, when the volume fraction ϕ increases, a rise in the thermal profile is observed. Figure 8 confirms this behavior for increasing values of volume fraction ϕ .

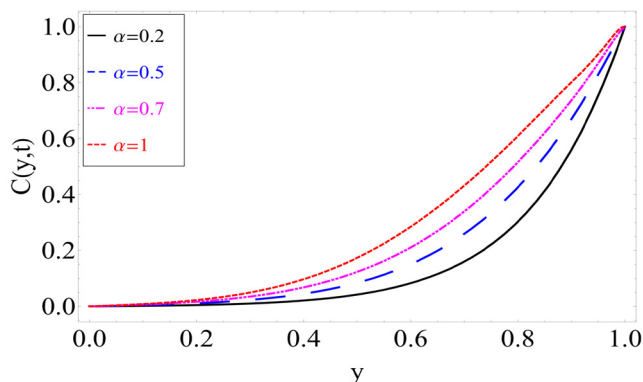


Fig. 9. Variation in concentration field against α

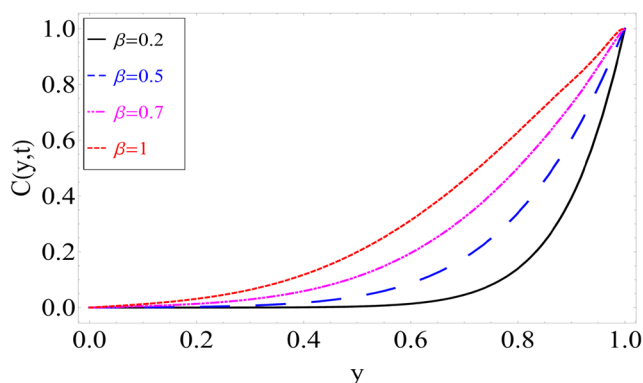


Fig. 10. Variation in concentration field against β

To examine how the volume fraction ϕ affects the concentration profile, Figure 11 is drawn. As the volume fraction ϕ value increases, the figure shows a rising concentration profile. The nanoparticles take up greater space in the matrix (distilled water) when the volume percentage of Al_2O_3 - TiO_2 - ZnO nanoparticles rises. A greater concentration of nanoparticles in the same volume of material is typically the result of this increased particle density. Therefore, a rising trend is observed in the figure for higher values of ϕ .

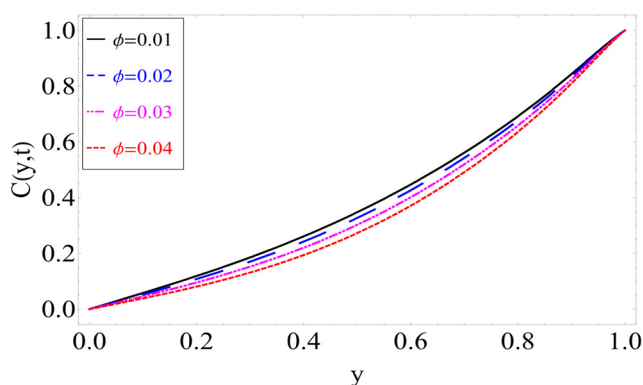


Fig. 11. Variation in concentration field against ϕ

The effect of Schmidt number Sc on the concentration profile is depicted in Figure 12. When compared to momentum diffusion, a high Schmidt value indicates noticeably slower mass diffusion. A drop in the fluid's concentration is seen because of the slower mass diffusion creating a steep concentration profile and a significant concentration gradient across the barrier.

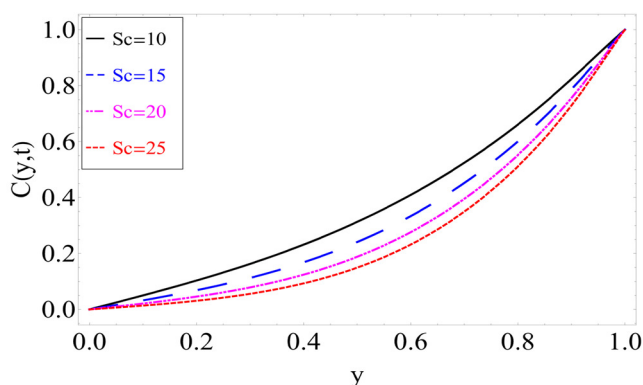


Fig. 12. Variation in concentration field against Sc

4. Conclusions

In the present study, a fractal-fractional order mathematical model for ternary hybrid nanofluid flow was developed and analyzed to examine the impact of key dimensionless parameters on velocity, temperature, and concentration profile. The Laplace transform technique was used to obtain exact solutions for parametric study.

The findings drawn from the present study contribute to a deeper understanding of the growing demands for enhanced heat transfer using nanofluid. The key conclusions are:

- i. The fractal-fractional model provides greater memory impact as compared to other models.
- ii. The fractal, fractional, and classical can easily be deduced from the existing model by adjusting the fractal and fractional order parameters.
- iii. The applied meshless technique is handy in dealing with fractal-fractional Navier Stokes thermal and mass diffusion equations.
- iv. The velocity profile of the ternary hybrid nanofluid increases with the parameters G_r and G_m , while decreases with higher values of ϕ .
- v. The temperature profile shows an increasing trend in the velocity profile for greater values of the Prandtl number ϕ .
- vi. The greater Schmidt number S_c enhances the concentration profile.

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Conflicts of Interest

The author declares no conflicts of interest.

References

- [1] Cox, R. G., & Mason, S. G. (1971). Suspended particles in fluid flow through tubes. *Annual Review of Fluid Mechanics*, 3(1), 291-316. <https://doi.org/10.1146/annurev.fl.03.010171.001451>.
- [2] Kahnert, M. (2016). Numerical solutions of the macroscopic Maxwell equations for scattering by non-spherical particles: a tutorial review. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 178, 22-37. <https://doi.org/10.1016/j.jqsrt.2015.10.029>.
- [3] Wang, X. Q., & Mujumdar, A. S. (2007). Heat transfer characteristics of nanofluids: a review. *International Journal of Thermal Sciences*, 46(1), 1-19. <https://doi.org/10.1016/j.ijthermalsci.2006.06.010>.
- [4] Wen, D., Lin, G., Vafaei, S., & Zhang, K. (2009). Review of nanofluids for heat transfer applications. *Particuology*, 7(2), 141-150. <https://doi.org/10.1016/j.partic.2009.01.007>.
- [5] Wang, J., Yang, X., Klemeš, J. J., Tian, K., Ma, T., & Sunden, B. (2023). A review on nanofluid stability: preparation and application. *Renewable and Sustainable Energy Reviews*, 188, 113854. <https://doi.org/10.1016/j.rser.2023.113854>.
- [6] Murtaza, S., Ali, F., Sheikh, N. A., Khan, I., & Nisar, K. S. (2021). Analysis of silver nanoparticles in engine oil: Atangana–Baleanu fractional model. *CMC-Computers, Materials & Continua*, 67(3), 2915-2932. <https://doi.org/10.32604/cmc.2021.013757>.
- [7] Murtaza, S., Ahmad, Z., Ali, I. E., Akhtar, Z., Tchier, F., Ahmad, H., & Yao, S. W. (2023). Analysis and numerical simulation of fractal-fractional order non-linear couple stress nanofluid with cadmium telluride nanoparticles. *Journal of King Saud University-Science*, 35(4), 102618. <https://doi.org/10.1016/j.jksus.2023.102618>.
- [8] Adun, H., Kavaz, D., & Dagbasi, M. (2021). Review of ternary hybrid nanofluid: Synthesis, stability, thermophysical properties, heat transfer applications, and environmental effects. *Journal of Cleaner Production*, 328, 129525. <https://doi.org/10.1016/j.jclepro.2021.129525>.
- [9] Muneeshwaran, M., Srinivasan, G., Muthukumar, P., & Wang, C. C. (2021). Role of hybrid-nanofluid in heat transfer enhancement—A review. *International Communications in Heat and Mass Transfer*, 125, 105341. <https://doi.org/10.1016/j.icheatmasstransfer.2021.105341>.

- [10] Waqas, H., Farooq, U., Liu, D., Abid, M., Imran, M., & Muhammad, T. (2022). Heat transfer analysis of hybrid nanofluid flow with thermal radiation through a stretching sheet: A comparative study. *International Communications in Heat and Mass Transfer*, 138, 106303. <https://doi.org/10.1016/j.icheatmasstransfer.2022.106303>.
- [11] Alghamdi, W., Alsubie, A., Kumam, P., Saeed, A., & Gul, T. (2021). MHD hybrid nanofluid flow comprising the medication through a blood artery. *Scientific Reports*, 11(1), 11621. <https://doi.org/10.1038/s41598-021-91183-6>.
- [12] Murtaza, S., Kumam, P., Sutthibutpong, T., Suttiarporn, P., Srisurat, T., & Ahmad, Z. (2024). Fractal-fractional analysis and numerical simulation for the heat transfer of ZnO+ Al₂O₃+ TiO₂/DW based ternary hybrid nanofluid. *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*, 104(2), e202300459. <https://doi.org/10.1002/zamm.202300459>.
- [13] Chen, W., Sun, H., & Li, X. (2022). Fractional derivative modeling in mechanics and engineering. New York, Springer Nature.
- [14] Ahmad, J., Ali, F., Murtaza, S., & Khan, I. (2021). Caputo time fractional model based on generalized Fourier's and Fick's laws for Jeffrey nanofluid: Applications in automobiles. *Mathematical Problems in Engineering*, 2021(1), 4611656. <https://doi.org/10.1155/2021/4611656>.
- [15] Asjad, M. I., Ikram, M. D., Ali, R., Baleanu, D., & Alshomrani, A. S. (2020). New analytical solutions of heat transfer flow of clay-water base nanoparticles with the application of novel hybrid fractional derivative. *Thermal Science*, 24(Suppl. 1), 343-350. <https://doi.org/10.2298/TSCI20S1343A>.
- [16] Ahmad, M., Asjad, M. I., & Singh, J. (2021). Application of novel fractional derivative to heat and mass transfer analysis for the slippage flow of viscous fluid with single-wall carbon nanotube subject to Newtonian heating. *Mathematical Methods in the Applied Sciences*. <https://doi.org/10.1002/mma.7332>.
- [17] Murtaza, S., Ali, F., Aamina, N. A., Sheikh, N. A., Khan, I., & Nisar, K. S. (2020). Exact analysis of non-linear fractionalized Jeffrey fluid. A novel approach of Atangana–Baleanu fractional model. *CMC-Computers, Materials & Continua*, 65(3), 2033-2047. <https://doi.org/10.32604/cmc.2020.011817>.
- [18] Murtaza, S., & Ahmad, Z. (2024). Analysis of Clay Based Cementitious Nanofluid Subjected to Newtonian Heating and Slippage Conditions with Constant Proportional Caputo Derivative. *GeoStruct Innovations*, 2(2), 53-67. <https://doi.org/10.56578/gsi020201>.
- [19] Murtaza, S., Ismail, E. A., Awwad, F. A., Bonyah, E., Hassan, A. M., Khan, M. S., et al. (2024). Parametric simulations of fractal-fractional non-linear viscoelastic fluid model with finite difference scheme. *AIP Advances*, 14(4). <https://doi.org/10.1063/5.0180414>.
- [20] Atangana, A. (2017). Fractal-fractional differentiation and integration: connecting fractal calculus and fractional calculus to predict complex system. *Chaos, Solitons & Fractals*, 102, 396-406. <https://doi.org/10.1016/j.chaos.2017.04.027>.
- [21] Murtaza, S., Ahmad, Z., Ali, I. E., Akhtar, Z., Tchier, F., Ahmad, H., & Yao, S. W. (2023). Analysis and numerical simulation of fractal-fractional order non-linear couple stress nanofluid with cadmium telluride nanoparticles. *Journal of King Saud University-Science*, 35(4), 102618. <https://doi.org/10.1016/j.jksus.2023.102618>.
- [22] Rehman, A., Inc, M., Mohammad, A., & Jan, R. (2025). Heat transfer in couple stress Williamson hybrid nanofluid with slip viscous dissipation and radiation and slip impacts: Semi-numerical simulation. *Modern Physics Letters B*, 2550182. <https://doi.org/10.1142/S0217984925501829>.
- [23] Khan, I., Alqahtani, A. M., Khan, A., Khan, D., Ganie, A. H., & Ali, G. (2022). New results of fractal fractional model of drilling nanoliquids with clay nanoparticles. *Fractals*, 30(01), 2250024. <https://doi.org/10.1142/S0218348X22500244>.
- [24] Zheng, K., Raza, A., Abed, A. M., Khursheed, H., Seddek, L. F., Ali, A. H., & Haq, A. U. (2023). New fractional approach for the simulation of (Ag) and (TiO₂) mixed hybrid nanofluid flowing through a channel: Fractal fractional derivative. *Case Studies in Thermal Engineering*, 45, 102948. <https://doi.org/10.1016/j.csite.2023.102948>.
- [25] Abbas, S., Ramzan, M., Shafique, A., Nazar, M., Metwally, A. S. M., Abduvalieva, D., et al. (2024). Analysis of heat and mass transfer on nanofluid: An application of hybrid fractal-fractional derivative. *Numerical Heat Transfer, Part A: Applications*. <https://doi.org/10.1080/10407782.2024.2366442>.
- [26] Zakian, V. (1969). Numerical inversion of Laplace transform, *Electronic Letters*, 5(6), 120-121. <https://doi.org/10.1049/el:19690090>.